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Toric Schemes over a Discrete Valuation Ring and Tropical Compactifications

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Toric Schemes over a Discrete Valuation Ring and Tropical Compactifications

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Let Y be a subvariety of an algebraic torus, Tevelv [24] defined and studied tropical compactifications as certain nice compactifications of Y. We give a criterion for certain compactification to be a schön compactification, and as a corollary, we show that any variety contains an open very affine schön variety. Using toric schemes defined over a discrete valuation ring, we generalize the theory of tropical compactification to the nonconstant coefficient case, i.e. for varieties defined over a discrete valuation ring.

Contents

Chapter 0

Introduction

In this thesis, we study compactfications of subvarieties of tori. Compactification problem plays an important role in algebraic goemetry, especially in moduli theory. This part of the story is traced back as early as Riemann's discovery; Riemann knew that the complex structure of Riemann surfaces of genus $g \geq 2$ depended on $3g - 3$ complex parameters, which suggests that M_g, the set of all algebraic curves of genus g is a variety of dimension $3g - 3$. Ever since then, study of properties of M_g (and other related moduli spaces) has been a central area in algebraic geometry.

It is awkward that M_g is not compact, most results in algebraic geometry are dealing with projective (or at least proper) varieties. Thus one hopes for a meaningful compactification of M_g. Mumford used his geometric invariant theory to get a nice compactification of M_g, which is the following:

Theorem 0.0.1 (Deligne-Mumford, [4])**.** *There is a coarse moduli space* $\overline{M}_g$

of stable curves of genus g, which contains M_g as open subvariety. $\overline{M}_g$ is projective.

It turns out that such compactification is not unique, for example in the case of M_g, there is a coarse moduli space of pseudo-stable curves, and in fact there are many other modular compactifications.

In the last few decades, another important developement in algebraic geometry is Mori theory, which attempts to classify algebraic varieties birationally. Mori theory however gives a canonical way to compactify open varieties (assuming some conjectures), we recall the result. Let X be a smooth open variety over an algebraically closed field of characteristic 0, then by Nagata's compactification theorem and Hironaka's resolution of singularity, we can compactifiy $X \subset \overline{X}$ such that $\overline{X}$ is proper and the boundary divisors $\overline{X} \backslash X$ are simple normal crossing.

The logarithmic m-pluricanonical forms $H^0(\overline{X}, m(K_{\overline{X}}+B))$ is independent of the smooth model $\overline{X}$, where $B = \overline{X} \backslash X$ is the boundary divisor. We form the logarithmic canoncal ring

$$R(X) = \bigoplus_{m \geq 0} H^0(\overline{X}, m(K_{\overline{X}} + B)).$$

A recent result shows that $R(X)$ is finitely generated over k (under some assumption).

Theorem 0.0.2 (see [2]). *If $(\overline{X}, \Delta)$ is a projective Kawamata log terminal* *pair, if* Δ *is big, and assume* $K_X + \Delta$ *is Cartier, then*

$$R(\overline{X}, K_X + \Delta) := \bigoplus_{m \geq 0} H^0(X, m(K_X + \Delta))$$

is finitely generated.

This theorem applies, in particular, to an open smooth variety X of log general type. If X is of log general type, then $\operatorname{Proj} R(X)$ is the log canoncal model of X, if moreover X is log minimal, meaning that some multiple of K_X gives an immersion of X, then the log canonical model of X can be thought of a canonical compactification of X.

It is natural to wonder if some modular compactifications of moduli spaces coincide with the log canonical compactification. This breaks down into two questions in usual. First, given a modular compactification, determine whether it is the log canonical compactification. Second, given a log canonical compactification, find a good modular meaning. It is a surprising result that the Deligne-Mumford compactification of M_g using stable curves is the log canonical compactification of M_g (see [3, 11] and references there).

In this thesis, we can determine whether a compactification is the log canonical compactification in some special cases. Our motivation comes from a study of the spaces $X(3, n)$, the moduli space of n labelled points on the projective plane $\mathbb{P}^2$ in linear position. There are known compactifications, for example Kapranov's Chow quotient compactification (see [14]). It is conjectured

Conjecture 0.0.3 (see [15])**.** *The Chow quotient compactification for* $X(3, n)$ *is the log canonical compactification for* $n = 6, 7, 8$*.*

This is proved for $n = 6$ (see [18]). It is observed that these varieties are very affine, meaning that they can be embedded into an algebraic torus, and they are log minimal (see [15]). A natural way to compactify a subvariety of a torus is to take its closure in various toric varieties containing the ambient torus. This was studied in a paper of J. Tevelev [24]. Our ultimate hope is that we could get the log canonical compactification in this way.

We say that a subvariety Y in a torus T is *schön* (as in [24]) if there is a toric variety X containing T, such that $\overline{Y}$, the closure of Y in X, is proper and the multiplication map

$$T \times \overline{Y} \to X$$

is smooth and surjective.

The surjectivity condition is to keep the support of the fan of X as small as possible. And the smoothness condition is to control the singularity of Y, in case Y is schön, $\overline{Y}$ has toroidal singularity, thus it could well be a candidate for the log canonical model. A weaker version only requires that the structure map

$$T \times \overline{Y} \to X$$

is flat and surjective, which is called a *tropical* compactification. It is a remarkable observation by Tevelev [24] that if $\overline{Y}$ is a tropical compactification, then the fan of the toric variety is supported on the tropicalization of Y, which explains our terminology.

Although a tropical compactification always exists, schön compactification does not necessarily exist. Existence of schön compactification is an intrinsic property of Y (thus we may call Y is schön), M. Luxton proved the following remarkable property of schön varieties, which in some sense further indicates the that schön property is an intrinsic property of Y.

Theorem 0.0.4 (Luxton, [18])**.** *If Y is a schön very affine variety, then any fan supported on the tropicalization of Y produces a schön compactification.*

We shall give a criterion (sufficient condition) to determine whether a given nice compactification of Y comes from a schön compactification. To describe the result, we first introduce some notations.

Y is a very affine smooth variety over an algebraically closed field, M_Y is the intrinsic lattice of units, i.e. $M_Y = \mathcal{O}^*(Y)/k^*$. Given $\overline{Y}$, a smooth compactification of Y, such that the boundary $\overline{Y}\backslash Y$ is a simple normal crossing divisor. For each stratum S, let M_S be the submomoid of M_Y consisting of all units which is regular on $\mathrm{Star}(S)$. Let σ_S be the dual cone of M_S in the dual space $N_Y = \mathrm{Hom}_{\mathbb{Z}}(M_Y, \mathbb{Z})$. Denote D_S to be the set of boundary divisors containing S.

Theorem 0.0.5. *Suppose following conditions are satisfied:*

1. *for any stratum S,* $\mathrm{Star}(S)$ *is affine and the ring homomophism*

$$k[M_S] \to \mathcal{O}(\mathrm{Star}(S))$$

is a surjection.

2. *for any stratum S, and any $D \in D_S$, there is a unit $u \in M_Y$ with valuation 1 along D and valuation* 0 *along other divisors in D_S.*

3. *the set of cones $\{\sigma_S\}$ form a fan in $N_Y \otimes_{\mathbb{Z}} \mathbb{R}$ when S runs over all strata.*

Then Y is schön, and there is a canonical embedding $\overline{Y} \to X(\Delta)$ preserving the stratification, with smooth and surjective structure map.

As an immediate application, we are able to show following result.

Theorem 0.0.6. *Any variety contains a schön open affine subvariety.*

Once we have a schön compactification, we are close to the log canonical model. We say Y is hübsch, if Y is schön and a schön compactification of Y is its log canonical compactification. It is known [9] that if Y is hübsch, then its tropicalization $\mathrm{trop}(Y)$ has a minimal fan structure.

Theorem 0.0.7. *If Y is a schön very affine variety, and assume the following assumptions are satisfied:*

1. *the tropicalization* $\mathrm{trop}(Y)$ *has a minimal fan structure Δ.*

2. *the schön compactification of Y corresponding to the minimal fan structure has irreducible intersection with each toric orbit of X of codimension* $< \dim Y$.

3. *for each cone $\sigma \in \Delta$,* $\mathrm{Star}(\sigma)$ *is not preserved by any translation.*

Then Y is hübsch.

As an application, we verify the following example.

Theorem 0.0.8. *An open cubic surface (a cubic surface minors it 27 lines) is hübsch, and its log canonical compactification is the cubic surface with all Eckhart points blown up. An Eckhart point of a cubic surface is an ordinary triple point of the 27 lines.*

We shall generalize the theory of tropical compactification to nonconstant coefficient case, that is for varieties defined over a discrete valuation ring. In that case, the tropicalization of Y is just a polyhedral complex (with certain properties of course), it may not be conical, therefore it may not have a fan structure.

To overcome this difficulty, we replace the tropicalization by the cone over it in a 1-dimensional larger space. This leads us to consider toric schemes over a discrete valuation ring. Let R be a discrete valuation ring, from an admissible relative fan Δ in $\widetilde{N}_{\mathbb{R}} = N_{\mathbb{R}} \oplus \mathbb{R}$ (admissible here means the the projection of Δ to the second factor lies in the positive half line), we can construct a toric scheme over $\operatorname{Spec} R$. A toric R-scheme $\mathfrak{X}$ contains torus T_N in the generic fiber and has an action

$$T_R \times_R \mathfrak{X} \to \mathfrak{X},$$

thus we can consider the closure of a subvariety of T_N in the toric R-scheme, and define tropical, schön, hübsch in a similar way. Results in the constant coefficient case have analoguous forms (with some modification) in the nonconstant coefficient case.

This thesis is organized as follows. We review tropical compactification in the constant coefficient case in chapter 1, and prove the criterion and existence of schön open subvarieties. In chapter 2, we study the construction and structure of toric R-schemes. We then generalize the theory of tropical compactification to nonconstant coefficient case in chapter 3. In chapter 4, we give some application and examples of tropical compactifications.

Chapter 1

Tropical Compactifications in Constant Coefficient Case

1.1 Tropicalization of a variety

1.1.1. We give a brief introduction to the tropicalization of a very affine variety in this section. Our notation and termilogy for toric varieties follows Fulton's book [5]. Let $\mathbb{K}$ be an algebraically closed field with a given non-trivial valuation $v : \mathbb{K}^* \to \mathbb{Q}$ such that $\mathbb{K}$ is complete with respect to this valuation. One may define tropicalization for any nontrivial valuation $v : \mathbb{K} \to \mathbb{R}$, but we don't seek the most generality here. See [21] for the more general setup. Our main example is $\mathbb{K} = \cup_{n\geq 1}\mathbb{C}((t^{1/n}))$, the Puiseux series over the complex numbers with the valuation given by the exponent of the lowest term. Let $T = \operatorname{Spec}\mathbb{K}[x_1^{\pm}, \ldots, x_n^{\pm}]$ be an algebraic torus over $\mathbb{K}$, with coordinates chosen, we can extend the valuation to $v : T(\mathbb{K}) \to \mathbb{Q}^n$.

Definition 1.1.2. For any closed subscheme Y of T, we define $\operatorname{trop}(Y)$, the *tropicalization* of Y to be the image of $Y(\mathbb{K}) \subset T(\mathbb{K})$ in $\mathbb{Q}^n$ under the valuation map v.

Theorem 1.1.3 ([21], 2.1.2). *Let $Y \subset T$ be a closed subscheme defined by an ideal $I \subset \mathbb{K}[x_1^{\pm}, \ldots, x_n^{\pm}]$, then the following sets are equal:*

1. $\operatorname{trop}(Y)$ *as defined in definition 1.1.2.*

2. *The set $\{(u(x_1), \ldots, u(x_n)) | u : \mathcal{O}(Y) \to \mathbb{Q} \cup \{\infty\}\}$, where u runs over all ring valuations $\mathcal{O}(Y) \to \mathbb{Q} \cup \{\infty\}$ extending v.*

3. *The set of $w \in \mathbb{Q}^n$ such that $\mathrm{in}_w f$ is not a monomial for any $f \in I \backslash \{0\}$, or equivalently $\mathrm{in}_w Y \neq \emptyset$.*

1.1.4. The second description in theorem 1.1.3 is usually called BG-set (short for Bieri-Grove set). Bieri and Grove [1] studied the set

$$\Delta_K^v(a_1, \ldots, a_n) := \{(u(a_1), \ldots, u(a_n)) | u : K^* \to \mathbb{R}\},$$

where $K \supset \mathbb{K}$ is any field extension, $a_1, \ldots, a_n \in K$ and u runs over all valuations $u : K^* \to \mathbb{R}$ which extend v. They showed that $\Delta_K^v(a_1, \ldots, a_n) \subset \mathbb{R}^n$ is a rational polyhedral complex of pure dimension equal to the transcendence degree of $K/\mathbb{K}$. When Y is integral, and apply Bieri-Grove's result on the function field of Y (i.e. consider $\Delta_{K(Y)}^v(x_1, \ldots, x_n)$), we conclude the following result.

Proposition 1.1.5. *Assume Y is integral, then* $\operatorname{trop}(Y)$ *is a rational polyhedral complex of pure dimension equal to* $\dim Y$*.*

1.1.6. Let k be an algebraically closed field, we take $\mathbb{K}$ to be the algebraic closure of $k(t)$ with induced valuation $v : \mathbb{K}^* \to \mathbb{Q}$ such that $v(t) = 1$. When $\operatorname{char} k = 0$, $\mathbb{K} = \cup_{n\geq 0} k((t^{1/n}))$ is the field of Puiseux series over k. For any closed subscheme Y of a torus T over k, we lift Y to $\mathbb{K}$ and define $\operatorname{trop}(Y)$ to be $\operatorname{trop}(Y \times_k \mathbb{K})$. We say that we are in the constant coefficient case. It is easy to see that if $w \in \operatorname{trop}(Y)$, then for any $a \in \mathbb{Q}_+$, $aw \in \operatorname{trop}(Y)$, thus $\operatorname{trop}(Y)$ is a conical polyhedral complex.

1.1.7. For the rest of this chapter we consider the constant coefficient case only, and we also assume Y is integral, i.e. Y is a subvariety of T over k. The intrinsic lattice of units of Y is $M_Y := \mathcal{O}^*(Y)/k^*$, which is a free abelian group of finite rank. When Y is a subvariety of T, $\mathcal{O}(Y)$ is generated as k-algebra by the characters of T, therefore it is also generated by M_Y. This gives an intrinsic embedding $Y \to \operatorname{Spec} k[M_Y] =: T_Y$, unique upto a scalar multiplication. For any other embedding of Y into a torus $Y \to T$, it factors through the intrinsic one:

$$\begin{tikzcd} Y \arrow[r] \arrow[dr] & T_Y \arrow[d, dashed] \\ & T. \end{tikzcd}$$

Definition 1.1.8. A variety Y is called *very affine* if it can be embedded in a torus. For a very affine variety Y, if no particular embedding is mentioned, we shall consider it via the intrinsic embedding, and $\operatorname{trop}(Y)$ is also the intrinsic one, i.e. $\operatorname{trop}(Y) \subset \operatorname{Hom}_{\mathbb{Z}}(M_Y, \mathbb{Q})$.

1.1.9. Let $a_1, \ldots, a_n \in \mathcal{O}^*(Y)$ be a basis of M_Y, then the intrinsic tropicalization $\operatorname{trop}(Y)$ is nothing but the BG-set

$$\operatorname{trop}(Y) = \Delta^v_{K(Y)}(a_1, \ldots, a_n),$$

where v is the trivial valuation on k. If we consider $\Delta^v_{K(Y)}(a_1, \ldots, a_n)$ sits inside $N_Y \otimes_{\mathbb{Z}} \mathbb{Q}$, then $\Delta^v_{K(Y)}(a_1, \ldots, a_n)$ does not depend on the choice of basis of $M_Y \otimes_{\mathbb{Z}} \mathbb{Q}$. This suggests a more general definition of tropicalization.

Definition 1.1.10. For any variety Y, not necessarily very affine, M_Y still makes sense and is a free abelian group of finite rank, then we define the *tropicalization* of Y to be

$$\operatorname{trop}(Y) := \Delta^v_{K(Y)}(a_1, \ldots, a_n) \subset N_Y \otimes_{\mathbb{Z}} \mathbb{Q},$$

for any $a_1, \ldots, a_n \in \mathcal{O}^*(Y)$ consisting of a basis of $M_Y \otimes_{\mathbb{Z}} \mathbb{Q}$.

1.1.11. When Y is very affine, the above definition agrees with the earlier definition of intrinsic tropicalization. In general we have a map $Y \to \operatorname{Spec}[M_Y]$, if Y' is the closure of the image of Y which is a very affine variety with intrinsic lattice M_Y, then $\operatorname{trop}(Y) = \operatorname{trop}(Y')$.

1.1.12. Tropicalization is functorial in the following sense. If $f : Y_1 \to Y_2$ is a morphism of any varieties, then there is an induced map of sets $f_* : \operatorname{trop}(Y_1) \to \operatorname{trop}(Y_2)$. If f is dominant, then f_* is a surjection. Indeed we have an induced map of units $f^*: M_{Y_2} \to M_{Y_1}$, hence a commutative diagram as follows,

$$\begin{array}{ccc} Y_1 & \longrightarrow & \operatorname{Spec} k[M_{Y_1}] \\ \downarrow & & \downarrow \\ Y_2 & \longrightarrow & \operatorname{Spec} k[M_{Y_2}]. \end{array}$$

Let Y_1' and Y_2' be the closure of image of Y_1 and Y_2 respectively, it is then clear we have an induced map

$$f_*: \operatorname{trop}(Y_1) = \operatorname{trop}(Y_1') \to \operatorname{trop}(Y_2') = \operatorname{trop}(Y_2).$$

If f is dominant, then $K(Y_1) \supseteq K(Y_2)$ is a field extension, we can choose a basis $a_1, \ldots, a_r$ of M_{Y_2} and $a_{r+1}, \ldots, a_n \in M_{Y_1}$ such that $f^*(a_1), \ldots, f^*(a_r), a_{r+1}, \ldots, a_n \in M_{Y_1}$ form a basis of $M_{Y_1} \otimes \mathbb{Z}\mathbb{Q}$. Since every valuation $w: K(Y_2)^* \to \mathbb{Q}$ can be extended to a valuation $w: K(Y_1)^* \to \mathbb{Q}$, we see that

$$\Delta^v_{K(Y_1)}(f^*(a_1), \ldots, f^*(a_r), a_{r+1}, \ldots, a_n) \to \Delta^v_{K(Y_1)}(a_1, \ldots, a_r)$$

is a surjection.

1.1.13. When Y is a smooth very affine variety, Hacking, Keel and Tevelev [9] observed that one can obtain $\operatorname{trop}(Y)$ from a compactification of Y with simple normal crossing boundary divisors. We shall prove a more general result, with Y being any smooth variety, $\operatorname{trop}(Y)$ being as in definition 1.1.10 and $Y \subset \overline{Y}$ being a toroidal embedding without self-intersection. For each boundary divisor D, the valuation val_D restricted on $M = M_Y$ determines a point in $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, still denoted by val_D. For each collection S of boundary divisors with nonempty intersection, let σ_S be the cone in N generated by val_D with $D \in S$, then we have the following theorem.

Theorem 1.1.14. $\mathrm{trop}(Y)$ *is the union of all* σ_S *where* S *runs over all collections of boundary divisors with nonempty intersection.*

1.1.15. We recall the main results of toroidal embeddings from [16]. If $Y \subset \overline{Y}$ is a toroidal embedding, this means that for any closed point $y \in \overline{Y}$, there exists a pointed affine toric variety (X_σ, x), such that locally analytically we have an isomorphism $\hat{\mathcal{O}}_{Y,y} \simeq \hat{\mathcal{O}}_{X_\sigma,x}$, which identifies the ideal of $\hat{\mathcal{O}}_{Y,y}$ generated by the ideal of the boundary $\overline{Y} - Y$ with the ideal of $\hat{\mathcal{O}}_{X_\sigma,x}$ generated by the toric boundary, then $\overline{Y} - Y$ has pure codimension 1. If the orbit of $x \in X_\sigma$ is closed, we say that (X_σ, x) is a local model of $y \in \overline{Y}$. If each irreducible boundary divisor $D \subset \overline{Y} - Y$ is normal, it is called a toroidal embedding without self-intersection.

1.1.16. We have a canonical stratification corresponding to $Y \subset \overline{Y}$ (a toroidal embedding without self-intersection). Let $\{D_1, \ldots, D_r\}$ be the set of irreducible boundary divisors, a stratum is defined to be an irreducible component of $\cap_{i \in I} D_i - \cup_{i \notin I} D_i$ for any $I \subset \{1, \ldots, r\}$. Mumford associates to each toroidal embedding (without self-intersection) a conical polyhedral complex with integral structure as follows. For each stratum S, let D_S be the set of boundary divisors containing S, and $\mathrm{Star}(S)$ be $\overline{Y} - \cup_{D_i \notin D_S} D$. Let M^S be the group of Cartier divisors of $\mathrm{Star}(S)$ supported on $\mathrm{Star}(S) \backslash Y$, M^S_+ be the submonoid of M^S consisting of effective Cartier divisors, $N^S = \mathrm{Hom}_{\mathbb{Z}}(M^S, \mathbb{Z})$.

Let σ^S be the cone in $N^S_{\mathbb{R}}$ spanned by $v \in N^S$ such that $\langle v, D\rangle \geq 0$ for any $D \in M^S_+$. For another stratum $S' < S$ (meaning S' is contained in the closure of S), we have a surjective map $M^{S'} \to M^S$ (restriction of divisors from $\mathrm{Star}(M^{S'})$ to $\mathrm{Star}(M^S)$), which induces a map $M^{S'}_+ \to M^S_+$ and corresponding maps $N^S \to N^{S'}$ and $\sigma^S \to \sigma^{S'}$. By gluing all $\sigma^S \subset N^S_{\mathbb{Q}}$, we get the associated (abstract) conical polyhedral complex $\Delta = (|\Delta|, \sigma^S, M^S)$.

1.1.17. The point for this is that locally $Y \subset \overline{Y}$ hehaves just like the toric variety associated to $\sigma^S \subset N^S_{\mathbb{Q}}$, although the vector spaces $N^S_{\mathbb{R}}$ are different for different strata. And for any point $y \in S \subset \overline{Y}$, $(X(\sigma^S), x_{\sigma^S})$ is the local model for y. The main theorem of toroidal embedding states that for any subdivision of Δ, there is another toroidal embedding $Y \subset \overline{Y}'$ and a canonical birational morphism $\overline{Y}' \to \overline{Y}$, such that the polyhedral complex associated to $\overline{Y}'$ is the subdivision of $\overline{Y}$. We are ready to prove theorem 1.1.14.

Proof of theorem 1.1.14. Y is very affine and $Y \subset \overline{Y}$ is a toroidal embedding. M_Y is the intrinsic lattice, we have a morphism $M \to M^S$ for any stratum S, namely $m \mapsto (m)$ on $\mathrm{Star}(S)$, hence a corresponding dual map $N^S \to N$ and $|\Delta| \to N_{\mathbb{Q}}$.

We claim that the image of $\sigma^S \subset N^S_{\mathbb{Q}}$ in $N_{\mathbb{Q}}$ is exactly $\sigma_S := \sigma_{D_S}$, thus the image of $|\Delta|$ is $\cup_S \sigma_S$ as in theorem 1.1.14. To see this, we note that the toric variety associated to $\sigma^S \subset N^S$ and the distinguished point of the closed orbit is a local model of $y \in S \subset \overline{Y}$. By [16] page 60 lemma 1,

$$M^S \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \bigoplus_{D \in D_S} \mathbb{Q} \cdot D,$$

and $M_+^S \otimes_{\mathbb{Z}} \mathbb{Q} = (\sigma^S)^\vee$ is generated by $D \in D_S$. The multiplicity map $\mathrm{mult}_D : M^S \to \mathbb{Z}$ is a one dimensional face of σ^S. Since $\langle \mathrm{mult}_D, (m) \rangle = \mathrm{val}_D\, m$ for any $m \in M_Y$, we see that mult_D is mapped to val_D, hence σ^S is mapped to σ_S.

Next we prove $\cup_S \sigma_S \subset \mathrm{trop}(Y)$. For any vector $w \in \sigma^S$, we can make a subdivision Δ' of Δ, such the the ray $\mathbb{R}_{\geq 0} w$ is a one dimensional face in Δ', by the main theorem of toroidal embeddings, there is another toroidal embedding $Y \subset \overline{Y}'$ such that the associated polyhedral complex is Δ', in particular, ther is a codimension one strata $S \subset \overline{Y}'$ corresponding to $\mathbb{R}_{\geq 0} w$, let D be the closure of S. The divisorial valuation val_D on M_Y lies exactly on the image of $\mathbb{R}_{\geq 0} \cdot w$.

Conversely, given $w \in \mathrm{trop}(Y)$, by (2) of theorem 1.1.3, a positive scalar multiple of w is of the form $u|_M$ for some valuation $u : \mathcal{O}(Z)^* \to \mathbb{Z}$ for some closed subvariety $Z \subset Y$, u then extends to a field valuation $u : K(Z)^* \to \mathbb{Z}$, let $R(Z)$ be the valuation ring, and clearly $R(Z) \supset \mathcal{O}(Z)$. We have the following commutative diagram,

$$\begin{tikzcd} \operatorname{Spec} K(Z) \arrow[r] \arrow[d] & Z \arrow[r] & \overline{Y} \arrow[d] \\ \operatorname{Spec} R(Z) \arrow[rr] \arrow[urr, dashed] & & \operatorname{Spec} k. \end{tikzcd}$$

Since $\overline{Y}$ is proper, by the valuative criterion, $\operatorname{Spec} R(Z) \to \operatorname{Spec} k$ factors through $\overline{Y}$. Assume the closed point of $\operatorname{Spec} R(Z)$ lands in the stratum $S \subset \overline{Y}$, we show that $w \in \sigma_S$. Indeed, the dual cone of σ_S in M_Y is

$$(\sigma_S)^\vee = \{m \in M_Y \,|\, \mathrm{val}_D\, m \geq 0, \forall D \in D_S\},$$

which is the same as

$$\{m \in M_Y | m \text{ extends to a regular functon on } \mathrm{Star}(S)\}.$$

For any $m \in (\sigma_S)^\vee$, m is also a regular function on Z, we have

$$\langle w, m \rangle \geq 0,$$

hence $w \in \sigma^S$.

□

1.2 Tropical Compactifications

1.2.1. Let $Y \subset T$ be a closed subvariety of a torus over k. Let M be the lattice of characters of T, and N be the lattice of one parameter subgroups of T, there is a natural nondegenerate pairing $M \times N \to \mathbb{Z}$. For any fan $\Delta \subset N_{\mathbb{Q}}$, we denote $X(\Delta)$ to be the normal toric variety associated to the fan Δ, and $\overline{Y} = \overline{Y}(\Delta)$ to be the closure of Y in $X(\Delta)$.

Definition 1.2.2 ([24]). We say $\overline{Y}$ is a *tropical compactification*, or Δ is a *tropical fan*, if $\overline{Y}$ is proper, and the structure map $T \times \overline{Y} \to X(\Delta)$ is flat and surjective.

Theorem 1.2.3 (Tevelev [24])**.** *1. If $Y \subset T$ is rigid, i.e. the stablizer of Y in T is trivial, then tropical compactification always exists.*

2. If Δ is a tropical fan, then Δ is supported on $\mathrm{trop}(Y)$*.*

3. If Δ is a tropical fan, then any refinement of Δ is also a tropical fan.

4. If Δ is a tropical fan, and $X(\Delta)$ is smooth (equivalently every cone in Δ is strictly simplicial), then $\overline{Y}$ is Cohen-Macauley at any 0-dimensional stratum.

1.2.4. It is not known if the rigidity assumption can be removed, the original proof of the existence of tropical compactification is a constructive proof. For any cone $\sigma \in \Delta$, and $x \in O_\sigma$, any closed point in the orbit corresponding to σ, the fiber of the structure map $T \times \overline{Y} \to X(\Delta)$ at x is isomorphic to $(\ker : T \to O_\sigma) \times (\overline{Y} \cap O_\sigma)$, where $\overline{Y} \cap O_\sigma$ is scheme-theoretic intersection. If Δ is a tropical fan, then fibers of the structure map are equidimensional, with dimension equal to $\dim Y$, thus $\overline{Y} \cap O_\sigma$ has pure codimension in $\overline{Y}$ equal to the codimension of O_σ in $X(\Delta)$. However the converse is not true, so it is generally hard to show whether a fan supported on $\operatorname{trop}(Y)$ is tropical or not. If $\overline{Y} \cap O_\sigma$ is reduced and has pure codimension in $\overline{Y}$ equal to the codimension of O_σ in $X(\Delta)$ for all $\sigma \in \Delta$, then Δ is a tropical fan. This is due to the following lemma.

Lemma 1.2.5. *If $f : X \to Y$ is a dominant morphism of a integral scheme into a normal scheme with reduced fibers of constant dimension, then f is flat.*

Proof. By [7] 14.4.4, f is universally open. Then by [7] 15.2.3, f is flat. □

Definition 1.2.6. Notations as in 1.2.2, we say $\overline{Y}$ is a *schön compactification*, or Δ is a *schön fan* if $\overline{Y}$ is a tropical compactification, and moreover the structure map $T \times \overline{Y} \to X(\Delta)$ is smooth.

Proposition 1.2.7 (Tevelev [24])**.** *If $Y \subset T$ admits a schön compactification, then any tropical fan is schön.*

1.2.8. If $\overline{Y}$ is a schön compactification, then $\overline{Y}$ intersects the toric boundary transversely, $\overline{Y} \cap O_\sigma$ is nonsingular, of pure codimension equal to the codimension of O_σ in $X(\Delta)$, and the converse is also true by lemma 1.2.5. An important result proved by Luxton about schön compactification is the following.

Theorem 1.2.9. *If $Y \subset T$ admits a schön compactification, then any fan Δ supported on* $\mathrm{trop}(Y)$ *is tropical, hence schön.*

1.2.10. Proof of theorem 1.2.9 will appear in next section since it is rather long and involved. The original proof in [18] is simplified and clarified by the author in a collaborative work [19], and is needed for generalizing to the nonconstant coefficient case. Note that in general, not every fan supported on $\mathrm{trop}(Y)$ is tropical. The following example is due to Sturmfels and Tevelev [23]. Let $X \subset \mathbb{P}^{N-1}$ be a projective variety, and $p \in X$ is a closed point such that X is not Cohen-Macauley at p. Assume $r = \dim X \geq 2$, we take $H_1, \ldots, H_r$ to be generic hyperplanes through p, and $H_{r+1}, \ldots, H_N$ to be generic hyperplanes, then $T = \mathbb{P}^{N-1}$ is a torus, let Y be $X \cap T$. The intersection of $\cap_{i \in I} H_i$ with X is nonempty if $|I| = r$ and empty if $|I| > r$, thus the subfan of the fan of $\mathbb{P}^{N-1}$ consisting of cones whose correspoingding orbits intersect X is supported on $\mathrm{trop}(Y)$ by [23] Proposition 3.9, but it cannot be a tropical fan by theorem 1.2.3 (4). The following result shows that if $Y \subset \overline{Y}$ is a schön compactification, then $\overline{Y}$ has good singularity.

Proposition 1.2.11. *If $\overline{Y} \subset X(\Delta)$ is a schön compactification, then $Y \subset \overline{Y}$ is a toroidal embedding (without self-intersection).*

Remark 1.2.12. It is often useful to consider an arbitrary Y, not necessarily irreducible. The reason is that if $\overline{Y}$ is a tropical (or schön) compactification of an irreducible variety Y, and $W \subset X(\Delta)$ is a toric orbit closure, then $\overline{Y} \cap W$ has flat (or smooth) surjective structure map in this toric variety, being the pullback of the structure map of $\overline{Y}$, but $\overline{Y} \cap O$ is usually not irreducible where $O \subset W$ is the open orbit (see also lemma 1.3.3). We can define tropical and schön compactification of an arbitrary closed scheme $Y \subset T$ in the same way as in definition 1.1.2. Note that it remains true in this general case that $|\Delta| = \mathrm{trop}(Y)$ when Δ is tropical, however some results are not true anymore, for example it is not clear if tropical compactification exists when Y is non-reduced. If Y is a closed subscheme and $\overline{Y}$ is a schön compactification, then Y is necessarily reducible. Since $T \times \overline{Y} \to X(\Delta)$ is smooth, we conclude that $\overline{Y}$ is a disjoint union of irreducible components. Let $\overline{Y}' \subset \overline{Y}$ be an irreducible component, then there is a possibly smaller toric open subset $X' \subset X(\Delta)$ containing $\overline{Y}'$ with smooth surjective structure map. It follows that the fan of X', being a subfan of Δ, is supported on $\mathrm{trop}(\overline{Y}' \cap T)$.

1.3 Proof of Theorem 1.2.9

1.3.1. Let $Y \subset T$ be schön, and Δ' any fan supported on $\mathrm{trop}(Y)$. Let Δ be a refinement of Δ' such that Δ is schön. Denote $\overline{Y}$ and $\overline{Y}'$ to be the closure of

Y in $X(\Delta)$ and $X(\Delta')$ respectively. We have a commutative diagram,

$$\begin{tikzcd}
\overline{Y} \arrow[r] \arrow[d] & X(\Delta) \arrow[d] & T \times_k \overline{Y} \arrow[l] \arrow[d] \\
\overline{Y}' \arrow[r] & X(\Delta') & T \times_k \overline{Y}' \arrow[l]
\end{tikzcd}$$

The first step is to show that the left square is a set-theoretic cartesian diagram, and so is the right square (theorem 1.3.5).

Lemma 1.3.2. *Let $Z \subset T$ be a Zariski closed subset of equidimension. If* $\mathrm{trop}(Z)$ *is preserved by translation by a linear subspace L of $N_{\mathbb{Q}}$, then each irreducible component of Z is preserved by the corresponding subtorus of T.*

Proof. Let $T \to T'$ be a homomorphism of tori corresponding to $N_{\mathbb{Q}} \to N_{\mathbb{Q}}/L$. Let $Z_i \subset Z$ be irreducible components, and Y_i the closure of the image of Z_i in T'. Since $\mathrm{trop}(Y_i)$ is the image of $\mathrm{trop}(Z_i)$ in $N_{\mathbb{Q}}/L$, we have $\dim \mathrm{trop}(Y_i) \leq \dim \mathrm{trop}(Z) - \dim L = \dim Z - \dim L$. On the other hand $\dim \mathrm{trop}(Y_i) = \dim Y_i \geq \dim Z_i - \dim L$, hence $\dim Z_i = \dim Y_i + \dim L$. Thus the generic fiber of $Z_i \to Y_i$ is the relative torus, hence $Z_i \cong Y_i \times \ker(T \to T')$. $\square$

Lemma 1.3.3. *Let $\overline{Y} \subset X(\Delta)$ be a tropical compatification, $W \subset X(\Delta)$ an orbit closure, and $Z = \overline{Y} \cap W$, then Z is equidimensional and each irreducible component intersects the open orbit in W.*

Proof. By pulling back the structure map, we see that the structure map of Z in W is also flat. By the openness of flat morphism, each irreducible component of Z intersects the open orbit O of W. Note that $(Z \cap O) \times \ker(T \to O)$ is the fiber of $T \times \overline{Y} \to X(\Delta)$ at any point in O, thus $Z \cap O$ is equidimensional and so is Z. $\square$

Lemma 1.3.4. *Notations and assumptions as in Lemma 1.3.3, if* $p : X(\Delta) \to X(\Delta')$ *is a proper toric map, then* Z *is preserved by* T_W *where* T_W *is the relative torus of* $W \to p(W)$.

Proof. Since $Z \subset W$ is a tropical compactification of $Z \cap O$, $\operatorname{trop}(Z \cap O)$ is the support of the fan of W (see remark 1.2.12), which is the inverse image of the support of the fan of $p(W)$ since p is proper. Thus $\operatorname{trop}(Z)$ is preserved by translation of a linear subspace of N_O whose corresponding subtorus of O is the relative torus T_W. It then follows from Lemma 1.3.2. $\square$

Theorem 1.3.5. *Let* $Y \subset T$ *be a closed subvariety,* $\overline{Y} \subset X(\Delta)$ *a tropical compactification,* $p : X(\Delta) \to X(\Delta')$ *a proper toric map,we have a commutative diagram*

$$\begin{tikzcd} \overline{Y} \arrow[r] \arrow[d] & X(\Delta) \arrow[d, "p"] \\ Y' \arrow[r] & X(\Delta'), \end{tikzcd}$$

where $Y' = p(\overline{Y})$*, then* $\overline{Y}$ *is a set-theoretic inverse image of* Y' *under the map* $X(\Delta) \to X(\Delta')$*. This applies in particular to the case when* Δ *is a refinement of* Δ'.

Proof. We show that if $y \in p^{-1}(y') \cap \overline{Y}$ then $p^{-1}(y') \subset \overline{Y}$. Let O' be the torus orbit containing y', $p^{-1}(O')$ is a union of toric varieties with connected fibres. It suffices to show that if y' falls in one irreducible component of $p^{-1}(O')$, then the whole fiber in that irreducible component is contained in $\overline{Y}$, then this follows from Lemma 1.3.4. $\square$

Corollary 1.3.6. *Notations as in Theorem 1.3.5, let P be the fiber product $Y' \times_{X(\Delta')} X(\Delta)$, then the induced map $\overline{Y} \to P$ is the reduction of P.*

Proof. P is a closed subscheme of X and so is Y, the induced map $\overline{Y} \to P$ is surjective by Theorem 1.3.5, since $\overline{Y}$ is integral, we have $\overline{Y} = P_{\text{red}}$. $\square$

The following lemma completes the proof of Theorem 1.2.9.

Lemma 1.3.7. *In the following diagram, X, X', Y and Y' are varieties with X, X' and Y normal, p is proper with connected fibers.*

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d] & X \arrow[d, "p"'] \\ Y' \arrow[r, "f'"] & X'. \end{tikzcd}$$

Let $P = Y \times_{X'} X$, assume the induced map $Y \to P$ is the reduction of P. If f is smooth, then so is f'.

Proof. Let $y' \in Y'(k)$ be a closed point, we show that f' is smooth at y'. Let $x' = f'(y')$, $F = Y'_{x'}$, the scheme-theoretic fibre over x' viewed as a closed subscheme of Y', and $G = X_{x'}$. Let $\widetilde{Y}$ be the normalization of Y, then the map $Y \to Y'$ factors through $\widetilde{Y}$, as we have a following diagram:

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d] & P \arrow[r, "j"] \arrow[d, "q"'] & X \arrow[d, "p"'] \\ \widetilde{Y} \arrow[r] & Y' \arrow[r, "f'"] & X'. \end{tikzcd}$$

Note that $j^{-1}(G) = F \times_k G$, and $f^{-1}(G)$ is a closed subscheme of $j^{-1}(G)$ with same support, and is smooth over G, thus we have $f^{-1}(G) = F_{\mathrm{red}} \times_k G$ and F_{red} is regular.

Let $f_1, \ldots, f_d \in \mathcal{O}_{F_{\mathrm{red}},y'}$ be a regular system of parameters, and lift them to the local ring $\mathcal{O}_{Y',y'}$ with the same notation. Assume $f_1, \ldots, f_d \in \Gamma(U, \mathcal{O}_{Y'})$, and we may shrink U a little to assume $U \cap V(f_1, \ldots, f_d) \cap F = \{y'\}$.

Let $U \to X' \times_k \mathbb{A}^d_k$ be the map defined by $(f', f_1, \ldots, f_d)$, and $\varphi : Y_U \to X \times_k \mathbb{A}^d_k$ defined by $(f, f_1, \ldots, f_d)$. For any point $y \in Y$ lying over y', $f_1, \ldots, f_d$ restricted in $\mathcal{O}_{f^{-1}(f(y)),y}$ is a regular system of parameters since $f^{-1}(f(y)) = F_{\mathrm{red}}$. By Lemma 1.3.8, φ is étale at y, thus φ is étale in a neighbourhood of Y'_y.

Since p has connected fibres, $\widetilde{Y} \to Y'$ is a homeomorphism of the underlying topological spaces, thus there is a unique $\tilde{y} \in \widetilde{Y}$ lying over y' and φ is also étale in a neighbourhood of $Y_{\tilde{y}}$. We have $\varphi^{-1}(X_x, 0) = Y_y$, so apply Lemma 1.3.9 for $W = Y_y \subset Y$ and $Z = (X_x, 0) \subset X \times_k \mathbb{A}^d_k$, and formal function theorem for proper maps $Y \to \widetilde{Y}$ and $X \times \mathbb{A}^d \to X' \times \mathbb{A}^d$, we have isomorphism of formal local rings $\widehat{\mathcal{O}}_{\widetilde{Y},\tilde{y}} \cong \widehat{\mathcal{O}}_{X' \times \mathbb{A}^d,(x,0)}$.

Thus $\widetilde{Y} \to X' \times \mathbb{A}^d$ is étale at $\tilde{y}$, so it separates tangent vectors at $\tilde{y}$. It follows that $\widetilde{Y} \to Y'$ also separates tangent vectors at $\tilde{y}$, hence $\widetilde{Y} \to Y$ is a closed immersion, we have $\widetilde{Y} \cong Y'$, and $f' : Y' \to X'$ is smooth.

$\square$

Lemma 1.3.8. *Let $f : Y \to X$ be a morphism of k-schemes, $y \in Y(k)$ a closed point, then f is smooth at y iff there exist $f_1, \ldots, f_d \in \mathfrak{m}_y \subset \mathcal{O}_{Y,y}$ such* *that the locally defined map $Y \to X \times_k \mathbb{A}^d_k$ given by $(f, f_1, \dots, f_d)$ is étale at y.*

Proof. If there is a map $Y \to X \times_k \mathbb{A}^d_k$ which is étale at y, by composing a smooth map $X \times_k \mathbb{A}^d_k \to X$, we see that $Y \to X$ is smooth at y. Conversely we know that Y_x is regular at y, choose $f_1, \dots, f_d$ in $\mathcal{O}_{Y,y}$ which form a regular system of parameters in $\mathcal{O}_{Y_x,y}$, consider the map

$$\varphi : \mathcal{O}_{X,x}[X_1, \dots, X_d]_{(X_1,\dots,X_d)} \to \mathcal{O}_{Y,y}$$

given by $X_i \mapsto f_i$. φ is flat since $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}[X_1, \dots, X_d]_{(X_1,\dots,X_d)}$ is faithfully flat, and φ is also geometrically regular since the geometric fiber of the closed point is a reduced point, thus φ is smooth of relative dimension 0, hence étale. □

Lemma 1.3.9. *Let $f : Y \to X$ be an étale morphism of schemes, $Z \subset X$ a closed subscheme, if $W = f^{-1}Z \to Z$ is an isomorphism, then $W_n \to Z_n$ is an isomorphism for all $n > 0$.*

Proof. Let I and J be the ideal sheaf of Z and W respectively. Clearly $J = f^{-1}I \cdot \mathcal{O}_Y$, and $J^n = f^{-1}I^n \cdot \mathcal{O}_Y$, thus $W_n = f^{-1}(Z_n)$ and we have a morphism $W_n \to Z_n$ for each $n > 0$ and it is an isomorphism for $n = 1$. It follows that $W_n \to Z_n$ is a homeomorphism for the underlying spaces, and is étale. Z is defined by a nilpotent ideal in Z_n, by the formal property of étale morphisms,

$$\begin{tikzcd} Z = W \arrow[r] \arrow[d] & W_n \arrow[d] \\ Z_n \arrow[r, "="] \arrow[ur, dashed] & Z_n, \end{tikzcd}$$

$W_n \to Z_n$ admits a section, then it follows that this is an isomorphism.

$\square$

1.4 Existence of Schön open subvariety

In this section, we prove the following theorem assuming $\operatorname{char} k = 0$.

Theorem 1.4.1. *Any variety over k contains a schön open dense very affine subvariety.*

1.4.2. This was conjectured in [24], where it proposed that this conjecture may be an alternative way of proving weak resolution theorem. However our proof only works in characteristic 0 and uses Hironaka's resolution theorem as an important step. It suffices to prove the theorem for a smooth variety Y. Our strategy is to first compactify Y so that the compactification $\overline{Y}$ is a smooth projective variety and the boundary $\overline{Y} \backslash Y$ is a simple normal crossing divisor. We then study how to embed $\overline{Y}$ into a toric variety X with one-to-one correspondence between strata of $\overline{Y}$ and toric strata of X and with a smooth structure map. Certain requirements have to be imposed on the log structure of Y (proposition 1.4.6). These requirements are achieved when we add more generic hyperplane sections to Y.

1.4.3. The problem of embedding an arbitrary variety (possibly singular and nonprojective) into some toric variety is studied in [27], and later in [12] for the equivariant case. Embedding a variety into a given toric variety is equivalent to giving a compatible log structure, namely the pullback of the natural log structure on the toric variety. Our proof has a similar flavor. We also notice that in [9], the authors obtained some similar requiements using quotient of affine conoid technique, but only applicable when the Picard group $\operatorname{Pic}\overline{Y}$ is a free abelian group of finite rank. We will indicate their relations in the sequel.

1.4.4. Let $\overline{Y}$ be a smooth proper variety over an algebraically closed field k of characteristic 0. Let $D = \bigcup_{i\in I} D_i$ be a simple normal crossing divisor. We have a stratification induced by D, a stratum is defined by $\bigcap_{j\in J} D_j - \bigcup_{i\in I\setminus J} D_i$ for a subset $J \subset I$, when $J = \emptyset$, the corresponding stratum is the open complement $Y = \overline{Y}\setminus D$. We don't require each stratum to be irreducible, this is a little different from the definition in [16] where a stratum is an irreducible component of our stratum here, but it is more convenient in our purpose for avoiding repeated cones in the geometric tropicalization. For a stratum S, let D_S be the set $\{D_i|S \subset D_i\}$, and let $\operatorname{Star}(S)$ be $Y - \bigcup_{D_i\notin D_S}$. Since D is simple normal crossing, a stratum S is regular (not necessarity irreducible), and also $\#D_S = \operatorname{codim} S$.

1.4.5. Fix (M,φ), a pair of an abstract lattice M and a group homomorphism $\varphi : M \to \mathcal{O}^*(Y)$ such that $k[M] \to \mathcal{O}(Y)$ is a surjection. For any submonoid $M' \subset M$, we have an induced k-algebra homomorphism $\varphi_* : k[M'] \to \mathcal{O}(Y)$. For any stratum S, let $M_S \subset M$ be the submonoid of M consisting all $m \in M$ such that m is regular on $\operatorname{Star}(S)$ or equivalently $\operatorname{val}_{D_i} m \geq 0$ for any $D_i \in D_S$. By abuse of notation sometimes we use m for which we actually mean $\varphi(m)$, for example $\operatorname{val}_D m$. Let N be the dual of M, for any stratum S, let $\sigma_S \subset N_{\mathbb{Q}}$ be the dual cone of M_S.

Proposition 1.4.6. *Notations as above, assume the following conditions are satisfied:*

1. *for any stratum S, $\mathrm{Star}(S)$ is affine and the induced map $k[M_S] \to \mathcal{O}(\mathrm{Star}(S))$ is surjective, in particular, Y is very affine.*
2. *for any stratum $S \neq Y$, and any $D_0 \in D_S$, there exists $m \in M$ such that $\mathrm{val}_{D_0} m = 1$ and $\mathrm{val}_{D_i} m = 0$ for any $D_i \in D_S - D_0$.*
3. *the collection of cones $\{\sigma_S\}$ as S runs over all strata is a fan Δ.*

Then there is a canonical closed immersion $Y \to X(\Delta)$ and $S = Y \cap O_{\sigma_S}$ as scheme-theoretic intersection for any stratum S, hence the structure map is smooth and surjective, $X(\Delta)$ is a schön compactification.

Remark 1.4.7. We explain our conditions and how to verify them practically. Condition 1 is automatic satisfied if $\mathrm{Star}(S)$ is very affine. Condition 2 would imply the intersection of $\overline{Y}$ and the toric variety is transversal as shown in the proof. It is equivalent to σ_S being strictly simplicial for all S. Condition 3 implies that the toric variety we build is separated, this condition is equivalent to the following: for any two strata S, S', their is a unit $m \in M$ such that $\mathrm{val}_D m > 0$ for all $D \in D_S \backslash D_{S'}$ and $\mathrm{val}_D m \leq 0$ for all $D \in D_{S'}$.

Proof. First note that σ_S is spanned by val_{D_i} for $D_i \in D_S$ since

$$M_S = \{m \in M | \, \mathrm{val}_{D_i} m \geq 0 \; \forall D_i \in D_S\}.$$

Condition (2) implies that val_{D_i} is part of a basis of N, i.e. σ_S is strictly simplicial.

For any stratum S, the surjection $k[M_S] \to \mathcal{O}(\mathrm{Star}(S))$ and the fact that $\mathrm{Star}(S)$ is affine determines a closed embedding $\mathrm{Star}(S) \to X_{\sigma_S}$. We put a partial ordering on the strata, we say $S_1 \leq S_2$ if $\mathrm{Star}(S_1) \subseteq \mathrm{Star}(S_2)$ or equivalently $D_{S_1} \subseteq D_{S_2}$ or equivalently $\sigma_{S_1} \leq \sigma_{S_2}$ (σ_{S_1} is a face of σ_{S_2}). Thus the correspondence $S \mapsto \sigma_S$ is an isomorphism of partially ordered sets. For any strata S_1, S_2, let S be the strata corresponding to $\sigma_{S_1} \cap \sigma_{S_2}$, we have a following commutative diagram,

$$\begin{tikzcd} \mathrm{Star}(S_1) \arrow[d] & \mathrm{Star}(S) \arrow[l] \arrow[r] \arrow[d] & \mathrm{Star}(S_2) \arrow[d] \\ X_{\sigma_{S_1}} & X_{\sigma_S} \arrow[l] \arrow[r] & X_{\sigma_{S_2}} \end{tikzcd}$$

By gluing schemes and morphisms, we get a canonical closed embedding $Y \to X(\Delta)$.

Next we prove $S = Y \cap O_{\sigma_S}$ as scheme-theoretic intersection. Assume $D_S = \{D_1, \ldots, D_l\}$. Let $E_i = D_i \cap \mathrm{Star}(S)$. By condition (2), we can find $m_1, \ldots, m_l$ such that $\mathrm{val}_{D_i} m_j = \delta_{ij}$. Note that m_i is part of a basis of M, the closed orbit $O_{\sigma_S} \subset X_{\sigma_S} = \mathrm{Spec}\, k[M_S]$ is defined by ideal $(m_1, \ldots, m_l) \subset k[M_S]$. Clearly $E_i = \mathrm{Star}(S) \cap (m_i = 0)$ since $(m_i) = E_i$, we have $S = \bigcap E_i = \mathrm{Star}(S) \cap (m_1 = \ldots = m_l = 0) = \mathrm{Star}(S) \cap O_{\sigma_S}$.

The structure map $T_M \times Y \to X(\Delta)$ has fiber $S \times \ker(T_M \to O_{\sigma_S})$ over any point $x \in O_{\sigma_S}$, so it has reduced, equidimensional, regular fiber. It follows that the structure map is surjective and smooth (Lemma 2.1.6).

$\square$

Remark 1.4.8. In [9], the authors assume that $\operatorname{Pic}\overline{Y}$ is a free abelian group of finite rank, and is generated by boundary divisors. They obtained the following criterion, a little different from ours. For each stratum S, let M^S be the sublattice of M_Y such that it has 0 valuations on divisors in D_S, thus we have a restriction map $M^S \to \mathcal{O}^*(S)/k^*$. If

1. for each stratum S is very affine and $M^S \to \mathcal{O}^*(S)$ is surjective,

2. same condition as Proposition 1.4.6 condition (2),

3. same condition as Proposition 1.4.6 condition (3),

then we have the same conclusion as in Proposition 1.4.6. They use affine conoid trick to show that one can embed $\overline{Y}$ into a toric variety with one-to-one correspondence of strata and smooth structure map. In this situation, condition (2) is also equivalent to the following: for any stratum S, $\operatorname{Pic}\overline{Y}$ is generated by boundary divisors not containing S.

In fact the assumption $\operatorname{Pic}\overline{Y}$ is free and generated by boundary divisors can be removed. Following the proof in [9], it still works assuming that the subgroup of $\operatorname{Pic}\overline{Y}$ generated by the boundary divisors is free. Let $\Lambda \subset \operatorname{Pic}\overline{Y}$ be the subgroup, then we use

$$W := \operatorname{Spec}_{\overline{Y}} \bigoplus_{L \in \Lambda} L \to \overline{Y}$$

in the proof, and it follows exactly the same way.

Here we provide another point of view, without even the assumption that Λ is free. Note that

$$M^S = \{m \in M_Y \,|\, \mathrm{val}_D\, m = 0, \forall D \in D_S\},$$

we have $M^S \subset M_S$ is the largest linear subspace contained in M_S. In other words $\mathrm{Spec}\, k[M^S]$ is the closed orbit in the affine toric variety $\mathrm{Spec}\, k[M_S]$. Clearly we have a map $k[M_S] \to \mathcal{O}(\mathrm{Star}(S))$ no matter $\mathrm{Star}(S)$ is affine or not, and therefore a map $\mathrm{Star}(S) \to \mathrm{Spec}\, k[M_S]$. We have a commuative diagram

$$\begin{tikzcd} \mathrm{Star}(S) \arrow[r] & \mathrm{Spec}\, k[M_S] \\ S \arrow[u] \arrow[r] & \mathrm{Spec}\, k[M^S]. \arrow[u] \end{tikzcd}$$

The new condition (1) implies that $S \to O_S := \mathrm{Spec}\, k[M^S]$ is a closed immersion and by condition (2), S is the scheme-theoretic inverse image of O_S. By glueing maps and morphisms, we still have a morphism $\overline{Y} \to X(\Delta)$ such that for each S, $S \to O_S$ is a closed immersion and S is the scheme-theoretic inverse image of O_S, thus it is a closed immersion. It then also implies our original condition (1).

In fact a weaker condition for new condition (1) is: (1*) each S is affine and $k[M^S] \to \mathcal{O}(S)$ is surjective. This is now equivalent to our old conditions, in other words conditions (1) (2) (3) are satisfied if and only if conditions (1*) (2) (3) are satisfied (which both imply that $\overline{Y}$ is canonically embedded in

$X(\Delta)$ with smooth structure map and one-to-one correspondence of strata), without any restriction on $\operatorname{Pic}\overline{Y}$.

Proof of Theorem 1.4.1. Let Y be a smooth variety, $\overline{Y}$ any smooth projective compactification of Y with simple normal crossing boundary divisor $D = \{D_1, \ldots, D_r\}$. Let L be a very ample line bundle such that $L + D_i$ is very ample for any i. Choose a finite set $E_i \subset |L + D_i|$ of generic sections with $\#E_i \geq \dim|L + D_i| + \dim Y + 1$ for $i = 0, 1, \ldots, r$ where we take D_0 to be the zero divisor. Let E be the union of all D_i and all divisors in E_i, $i = 0, 1, \ldots, r$. By Bertini's theorem, E is simple normal crossing. Let Y^o be the complement $\overline{Y}\backslash E$ and $M \cong \mathcal{O}(Y^o)/k^*$. We show that $(\overline{Y}, E, M)$ satisfies the assumptions in proposition 1.4.6.

The following facts are frequently used: if L is a very ample line bundle on a projective variety Y, and $s_1, \ldots, s_l$ are sections of $|L|$ in linear general position with $l = \dim|L| + 1$, then the complement is very affine. Intersection of two very affine open subvarieties of Y is again very affine, hence in the first statement, it is true for any $l \geq \dim|L| + 1$.

Let S be any stratum, we have $\#D_S \leq \dim Y$. For any $D_i \in D\backslash D_S$, pair with a divisor $D_i' \in E_0\backslash D_S$. We can write

$$E\backslash D_S = \bigcup_{i=0}^{r} F_i,$$

where $F_i = E_i\backslash D_S$ or $E_i\backslash D_S \cap \{D_i + D_i'\}$ if $D_i \in D\backslash D_S$. Each $F_i \subset |L + D_i|$ are sections in linear general position with $\#F_i \geq \dim|L + D_i| + 1$, thus

$\mathrm{Star}(S) = \cap(\overline{Y}\backslash F_i)$ is very affine. Since $\mathcal{O}(\mathrm{Star}(S))$ is generated by all units, it is also generated by M_S which contains all the units on $\mathrm{Star}(S)$.

To see condition (2), let $F \in D_S$ be any divisor, if F is one of D_i's, choose some $F' \in F_0\backslash D_S$ and consider $F+F' \in F_i$, otherwise $F \in F_i$ for some i, we choose another divisor $G \in F_i\backslash D_S$ and $G \neq F$ or $F+F'$. There is a unit $m \in M$ with the associated divisor $(m) = F - G$ or $F + F' - G$. In either case, this m satisfies condition (2).

To verify condition (3), we show that for any two strata S, S', there is a unit $m \in M$ such that $\mathrm{val}_F\, m > 0$ for all $F \in D_S\backslash D_{S'}$ and $\mathrm{val}_F\, m \leq 0$ for all $F \in D_{S'}$. Indeed for each $F \in D_S\backslash D_{S'}$, as in the above argument we can find $m_F \in M$ such that $(m_F) = F - G$ (or $F + F' - G$ if $F \in D$, paired with some $F' \in F_0\backslash D_{S'}$) and $G \notin D_S$. The product of all m_F will do. Thus $\sigma_S \cap \sigma_{S'}$ is their common face, condition (3) is satisfied. □

We state a sufficient condition to obtain a hübsch very affine variety to close this section. It is known that if $Y \subset T$ is a closed subvariety and is schön, then Y is either log minimal, or Y is preserved by a nontrivial subtorus of T (see [9], Theorem 3.1). If $\overline{Y}$ is a schön compactification, then $\overline{Y}$ is the log canonical compactification if and only if each strata $S \subset \overline{Y}$ is log minimal ([9], Theorem 9.1). And Y is preserved by a subtorus if and only if $\mathrm{trop}(Y)$ is preserved by a linear subspace of $N_{\mathbb{Q}}$ corresponding to the subtorus (Lemma 1.3.2). If $\overline{Y}(\Delta) \subset X(\Delta)$ is a schön compactification, then $\overline{Y} \cap V_\sigma$ also have smooth structure map in the toric variety V_σ, hence it is a schön compactification of $\overline{Y} \cap O_\sigma$. We should be careful that $\overline{Y} \cap O_\sigma$ may not be irreducible, therefore $\mathrm{trop}(\overline{Y} \cap O_\sigma)$

(which is $\mathrm{Star}(\sigma)$ following Fulton's notation [5]) being not preserved by a linear subspace does not imply that each irreducible component of $\overline{Y} \cap O_\sigma$. In conclusion, we obtain the following theorem.

Theorem 1.4.9. *Let Y be a very affine schön variety. Assume the following assumptions are satisfied:*

1. $\mathrm{trop}(Y)$ *has a minimal fan structure* Δ*,*

2. *for each cone* $\sigma \in \Delta$*,* $\overline{Y} \cap O_\sigma$ *is reduced,*

3. *for each* $\sigma \in \Delta$*,* $\mathrm{Star}(\sigma)$ *is not preserved by any translation,*

then Y is hübsch.

Chapter 2

Toric Schemes over a Discrete Valuation Ring

2.1 Construction from a relative fan

2.1.1. We fix following notations. R is a discrete valuation with quotient field K and residue field k. t is a fixed uniformizer and $\mathfrak{m}$ is the maximal ideal. Denote the generic point of $\operatorname{Spec} R$ by η and special point by s. For any scheme X over $\operatorname{Spec} R$, we write X_η and X_s for the fiber over η and s respectively. In this chapter, T is a torus over $\operatorname{Spec} \mathbb{Z}$, i.e. $T \simeq \operatorname{Spec} \mathbb{Z}[t_1^{\pm}, \ldots, t_n^{\pm}]$. First we give a formal definition of a toric scheme over $\operatorname{Spec} R$

Definition 2.1.2. A (normal) toric scheme over $\operatorname{Spec} R$ is an integral normal scheme $\mathfrak{X}$ together with a map $\mathfrak{X} \to \operatorname{Spec} R$ which is separated, and of finite type, such that it contains the torus T_K in its generic fibre $\mathfrak{X}_\eta$ and there is a group scheme action $T_R \times_R \mathfrak{X} \to \mathfrak{X}$ which extends the left multiplication of T_K on itself.

2.1.3. We give a first construction of toric R-scheme in this section via a relative fan. Let N be a lattice (i.e. a free abelian group of finite rank) and $\widetilde{N}$ be $N \oplus \mathbb{Z}$, then we have a canonical short exact sequence,

$$0 \longrightarrow N \longrightarrow \widetilde{N} \xrightarrow{p} \mathbb{Z} \longrightarrow 0.$$

Let $\widetilde{N}_{\mathbb{Q}}^{+}$ be the positive half space $p^{-1}(\mathbb{Q}_{\geq 0})$, and we have a duel exact sequece,

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{M} \longrightarrow M \longrightarrow 0.$$

Let $\sigma \subset \widetilde{N}_{\mathbb{Q}}^{+}$ be any cone, and $\sigma^{\vee}$ its dual in $\widetilde{M}_{\mathbb{Q}}$. Denote e to be the image of 1 in $\widetilde{M}$ under the map $\mathbb{Z} \to \widetilde{M}$, i.e. $e = (0, 1)$ under the decomposition $\widetilde{M} = M \oplus \mathbb{Z}$. Clearly $e \in \sigma^{\vee}$ since σ is contained in the positive half space. We write a lattice point of $\widetilde{M}$ in the form $\widetilde{m} = (m, r)$ for $m \in M$ and $r \in \mathbb{Z}$. For any submonoid $S \subset \widetilde{M}$ with $e \in S$, we define

$$A[S] := R[\chi^m t^r]_{\widetilde{m} \in S},$$

and define

$$\mathfrak{X}_\sigma := \operatorname{Spec} A[\sigma^{\vee} \cap \widetilde{M}].$$

Proposition 2.1.4. *$\mathfrak{X}_\sigma$ is an affine toric R-scheme in the sense of definition 2.1.2.*

Before proving this proposition, we first introduce some notations. For any $\sigma \in \widetilde{N}_{\mathbb{Q}}^{+}$, let $\bar{\sigma} := \sigma \cap N_{\mathbb{Q}}$, which is a face of σ. We use $\bar{\sigma}^{\vee}$ to mean the dual cone in $M_{\mathbb{Q}}$, note that $\bar{\sigma}^{\vee}$ is the image of $\sigma^{\vee}$ via the map $\widetilde{M}_{\mathbb{Q}} \to M_{\mathbb{Q}}$. More generally for any fan $\Delta \subset \widetilde{N}_{\mathbb{Q}}^{+}$, we denote

$$\overline{\Delta} := \{\bar{\sigma} | \sigma \in \Delta\},$$

which is a subfan of Δ.

Definition 2.1.5. A fan $\Delta \subset \widetilde{N}_{\mathbb{Q}}$ or a cone $\sigma \subset \widetilde{N}_{\mathbb{Q}}$ is called *admissible* if it is contained in $\widetilde{N}_{\mathbb{Q}}^{+}$.

Proof of Lemma 2.1.4. $\mathfrak{X}_\sigma$ is integral, of finite type and separated over $\operatorname{Spec} R$ and normal by the following lemma 2.1.6.

$$\mathfrak{X}_\sigma \times_R K = \operatorname{Spec} K[\bar{\sigma}^{\vee} \cap M] \supset \operatorname{Spec} K[M] = T_K,$$

i.e. the generic fiber of $\mathfrak{X}_\sigma$ contains a torus T_K.

The action $\mu : T_R \times_R \mathfrak{X}_\sigma \to \mathfrak{X}_\sigma$ comes from the dual action of algebras

$$\mu^* : A[\sigma^{\vee} \cap \widetilde{M}] \to R[M] \otimes_R A[\sigma^{\vee} \cap \widetilde{M}]$$

defined by

$$\chi^m t^r \mapsto \chi^m \otimes \chi^m t^r.$$

Note that this is compatible with the multiplication of T_K, i.e. $K[M] \to K[M] \otimes_K K[M]$.

To check this is an action, let

$$\lambda^* : R[M] \to R[M] \otimes_R R[M],\ \chi^m \mapsto \chi^m \otimes \chi^m$$

denote the multiplication $\lambda : T_R \times_R T_R \to T_R$, we see that the following diagram is commutative,

$$\begin{tikzcd}
A[\sigma^\vee \cap \widetilde{M}] \arrow[r, "\mu^*"] \arrow[d, "\mu^*"] & R[M] \otimes_R A[\sigma^\vee \cap \widetilde{M}] \arrow[d, "(\mathrm{id},\mu^*)"'] \\
R[M] \otimes_R A[\sigma^\vee \cap \widetilde{M}] \arrow[r, "(\lambda^*,\mathrm{id})"] & R[M] \otimes_R R[M] \otimes_R A[\sigma^\vee \cap \widetilde{M}]
\end{tikzcd}$$

□

Lemma 2.1.6. $A[\sigma^\vee \cap \widetilde{M}]$ *is a normal, finitely generated integral R-algebra.*

Proof. $\sigma^\vee \cap \widetilde{M}$ is a finitely generated monoid and $A[\sigma^\vee \cap \widetilde{M} \subset K[M]$ is a subring, hence it is integral and finitely generated over R.

To show it is normal, since $A[\sigma^\vee \cap \widetilde{M} \subset K[M]$ are both M-graded, and $K[M]$ is integral, it suffices to show that a homogeneous element of $K[M]$, if integral over $A[\sigma^\vee \cap \widetilde{M}]$, is contained in $A[\sigma^\vee \cap \widetilde{M}]$. Assume $a\chi^m \in K[M]$ with $a \in K^*, m \in M$ satisfies a monic polynormial

$$f(X) = X^l + c_1 X^{l-1} + \cdots + c_l$$

with coefficients $c_i \in A[\sigma^\vee \cap \widetilde{M}]$. Then we have

$$a^l \chi^{lm} + c_1 a^{l-1} \chi^{(l-1)m} + \cdots + c_l = 0.$$

If we only look at the χ^{lm} homogeneous part of the above equation, we may assume $c_i = b_i\chi^{im}$ are homogeneous where $b_i \in K$. Therefore we have

$$a^l + b_1 a^{l-1} + \cdots + b_l = 0,$$

and for some $i \geq 1$, $v(a^l) \geq v(b_i a^{l-i})$. From $iv(a) \geq v(b_i)$, we obtain $(a\chi^m)^i \in A[\sigma^\vee \cap \widetilde{M}]$. Since $\sigma^\vee \cap \widetilde{M}$ is saturated, $a\chi^m \in A[\sigma^\vee \cap \widetilde{M}]$. $\square$

Lemma 2.1.7. *Any normal affine toric R-scheme $\mathfrak{X}$ is of the form $\mathfrak{X} = \mathfrak{X}_\sigma$.*

Proof. Assume $\mathfrak{X} = \operatorname{Spec} A$, since torus T_R acts on $\operatorname{Spec} A$, by the diagonalizability of T, A is M-graded, i.e. $A = \bigoplus_{m \in M} A_m$. There are three possibilities,

$$A_m = 0,$$

or

$$A_m = K \cdot \chi^m,$$

or

$$A_m = R \cdot t^r \chi^m,$$

for some $r \in \mathbb{Z}$. The set $S = \{\widetilde{m} = (m, r) | \chi^m t^r \in A\}$ is a finitely generated saturated monoid since A is finitely generated and normal. And the quotient group of S is M since it contains $\operatorname{Spec} K[M]$ on the generic fiber, moreover $e \in S$ thus S is of the form $\sigma^\vee \cap \widetilde{M}$ for some cone $\sigma \in \widetilde{N}_{\mathbb{Q}}^{+}$.

$\square$

2.1.8. If $\tau < \sigma$ is a face, then the containment $\tau^\vee \cap \widetilde{M} \supset \sigma^\vee \cap \widetilde{M}$ induces a

T_R-equivariant open immersion $\mathfrak{X}_\tau \subset \mathfrak{X}_\sigma$. Let Δ be an admissible fan, then we can glue $\mathfrak{X}_\sigma : \sigma \in \Delta$ together to form a scheme $\mathfrak{X}(\Delta)$ as follows: if σ, τ are two cones in Δ, let $\delta = \sigma \cap \tau$ be their common face, then glue $\mathfrak{X}_\sigma$ and $\mathfrak{X}_\tau$ along their common open subset $\mathfrak{X}_\delta$.

Lemma 2.1.9. $\mathfrak{X}(\Delta)$ *is a normal toric R-scheme in the sense of definition 2.1.2.*

Proof. Since $\mathfrak{X}(\Delta) = \cup_{\sigma\in\Delta}\mathfrak{X}_\sigma$ with open dense subset T_K, $\mathfrak{X}(\Delta)$ is irredicible, reduced, normal, and is of finite type over $\operatorname{Spec} R$. The gluing is equivariant, so the T_R-action extends to $\mathfrak{X}(\Delta)$. The only thing it remains to prove is separatedness.

We need to show

$$\mathfrak{X}(\Delta) \to \mathfrak{X}(\Delta) \times_R \mathfrak{X}(\Delta)$$

is a closed immersion. It reduces to show that

$$\mathfrak{X}_\delta \to \mathfrak{X}_\sigma \times_R \mathfrak{X}_\tau$$

is a closed immersion, where δ is the common face of σ and τ. The latter is equivalent to the surjectivity of the homomorphism of corresponding rings

$$A[\sigma^\vee \cap \widetilde{M}] \otimes_R A[\tau^\vee \cap \widetilde{M}] \to A[\delta^\vee \cap \widetilde{M}],$$

given by $t^r\chi^m \otimes t^{r'}\chi^{m'} \to t^{r+r'}\chi^{m+m'}$. This follows from the fact that

$$(\sigma^\vee \cap \widetilde{M}) + (\tau^\vee \cap \widetilde{M}) = \delta^\vee \cap \widetilde{M}.$$

$\square$

Remark 2.1.10. Suppose R contains the residue field k, then there is an equivalent description of the above construction. If Δ is an admissible fan, the map of fans $\Delta \to \mathbb{Q}^{\geq}$ induces an equivariant map of ordinary toric varieties over field k,

$$X(\Delta) \to \mathbb{A}^1_k.$$

Then $\mathfrak{X}(\Delta)$ is the pull back of $X(\Delta)$ via $\operatorname{Spec} R \to \mathbb{A}^1_k$, where the map is defined by homomorphism of k-algebras

$$k[x] \to R : x \mapsto t.$$

That is to say

$$\mathfrak{X}(\Delta) = X(\Delta) \times_{\mathbb{A}^1_k} \operatorname{Spec} R.$$

To see this, note that $X(\Delta)$ is contructed by gluing open affine toric varieties X_σ in the same way. We only need to check that

$$X_\sigma \times_{\mathbb{A}^1_k} \operatorname{Spec} R = \mathfrak{X}_\sigma.$$

This is done by checking the corresponding rings. Since

$$k[\sigma^\vee \cap \widetilde{M}] \otimes_{k[x]} R = R[\sigma^\vee \cap \widetilde{M}]/(\chi^e - t) = A[\sigma^\vee \cap \widetilde{M}],$$

the statement follows.

Remark 2.1.11. It is worth pointing out that the toric R-scheme $\mathfrak{X}(\Delta)$ only depends on the exact sequence

$$0 \to N \to \widetilde{N} \to \mathbb{Z} \to 0,$$

and does not depend on the splitting $\widetilde{N} \simeq N \oplus \mathbb{Z}$. This is clear from the pull back description above. In general, a splitting only makes it convenient to write down the explicit form of $\mathcal{O}(\mathfrak{X}_\sigma)$, i.e $A[\sigma^\vee \cap \widetilde{M}]$, a different splitting gives isomorphic R-algebra in different form though. This would be useful when we consider tropical compactification in the non-constant coefficient case.

2.1.12. Toric R-schemes share many analogous properties of ordinary toric varieties only with minor changes in the proof, however it is not easily accessible in the literature, so we give some brief proof, indicating some modifications.

Proposition 2.1.13. *There is a bijection between orbits of T_K in $\mathfrak{X}(\Delta)_\eta$ and cones of Δ in $N_{\mathbb{Q}}$, and orbits of T_k in $\mathfrak{X}(\Delta)_s$ and cones of Δ not contained in $N_{\mathbb{Q}}$. Denote O_σ for the orbit corresponding the σ, and V_σ for the closure of O_σ, then $\tau \leq \sigma$ iff $V_\tau \supseteq O_\sigma$, and* $\dim \sigma$ *is equal to the codimension of O_σ in $\mathfrak{X}(\Delta)$.*

Proof. Since $\mathfrak{X}(\Delta)$ is obtained by glueing $\mathfrak{X}_\sigma$ together equivariantly, it suffices to show the proposition for $\mathfrak{X}_\sigma$. For any $\tau \subset \widetilde{N}^+_{\mathbb{Q}}$, we define

$$O_\tau = \operatorname{Spec} A[\tau^\perp \cap \widetilde{M}]$$

if $\tau \subset N_{\mathbb{Q}}$ or

$$O_\tau = \operatorname{Spec} k[\tau^\perp \cap \widetilde{M}]$$

if τ is not contained in $N_{\mathbb{Q}}$. Note that in the first case O_τ is a torus over K and in the second case, it is a torus over k. If $\tau < \sigma$ is a face, to realize O_τ as an orbit in $\mathfrak{X}_\sigma$, we embed O_τ into $\mathfrak{X}_\sigma$ via the corresponding ring homomorphism

$$A[\sigma^\vee \cap \widetilde{M}] \to A[\tau^\perp \cap \widetilde{M}]$$

in the first case, or

$$A[\sigma^\vee \cap \widetilde{M}] \to k[\tau^\perp \cap \widetilde{M}]$$

in the second case, by sending $t^r\chi^m$ to 0 if $\widetilde{m} = (m, r) \notin \tau^\perp$. Thus we equivariantly embeds O_τ into $\mathfrak{X}_\sigma$ as an orbit. The closure V_τ is given by

$$\operatorname{Spec} A[\tau^\perp \cap \sigma^\vee \cap \widetilde{M}]$$

or

$$\operatorname{Spec} k[\tau^\perp \cap \sigma^\vee \cap \widetilde{M}]$$

for $\tau \subset N_{\mathbb{Q}}$ or $\tau \not\subset N_{\mathbb{Q}}$ respectively. Everything else is clear. $\square$

2.1.14. The orbit closure V_σ is a normal toric variety over k (if $\sigma \not\subset N_{\mathbb{Q}}$) or a toric R-scheme (if $\sigma \subset N_{\mathbb{Q}}$). The fan of V_σ is given by $\mathrm{Star}(\sigma)$ (our terminology here follows [5]). $\mathrm{Star}(\sigma)$ sits in the quotient space of $\widetilde{N}$ modulo the linear space spanned by σ, and if $\sigma \subset N_{\mathbb{Q}}$, the quotient space has induced map to $\mathbb{Z}$, then $V_\sigma = X(\mathrm{Star}(\sigma))$ (if $\sigma \not\subset N_{\mathbb{Q}}$), or we have $V_\sigma = \mathfrak{X}(\mathrm{Star}(\sigma))$ (if $\sigma \subset N_{\mathbb{Q}}$). From proposition 2.1.13, we see that the generic fiber of $\mathfrak{X}(\Delta)$ is $X(\Delta \cap N_{\mathbb{Q}})$.

2.1.15. Let $\widetilde{N}$ and $\widetilde{N}'$ be two lattices which satisfy short exact sequences

$$0 \longrightarrow N \longrightarrow \widetilde{N} \xrightarrow{p} \mathbb{Z} \longrightarrow 0$$

and

$$0 \longrightarrow N' \longrightarrow \widetilde{N}' \xrightarrow{p'} \mathbb{Z} \longrightarrow 0$$

respectively. Let $\phi : \widetilde{N} \to \widetilde{N}'$ be a homomorphism of lattices which commutes with two projections,

$$\begin{tikzcd} \widetilde{N} \arrow[r, "\phi"] \arrow[dr, "p"'] & \widetilde{N}' \arrow[d, "p'"] \\ & \mathbb{Z}. \end{tikzcd}$$

Let Δ and Δ' be two admissible fans in $\widetilde{N}_{\mathbb{Q}}$ and $\widetilde{N}'_{\mathbb{Q}}$ respectively, such that ϕ induces a fan map $\phi : \Delta \to \Delta'$, i.e. ϕ maps cones of Δ into cones of Δ', then ϕ also induces an equivariant morphism of R-schemes

$$\phi_* : \mathfrak{X}(\Delta) \to \mathfrak{X}(\Delta').$$

Indeed ϕ induces a homomorphism of dual spaces

$$\begin{tikzcd} \widetilde{M} & \widetilde{M'} \arrow[l, "\phi^*"'] \\ & \mathbb{Z}. \arrow[ul] \arrow[u] \end{tikzcd}$$

If $\sigma \in \Delta$ and $\sigma' \in \Delta'$ are two cones such that $\phi(\sigma) \subset \sigma'$, then we have

$$\phi^* : \sigma'^{\vee} \to \sigma^{\vee}.$$

Note that $e' \mapsto e$, we thus have a homomorphism of R-algebras

$$A[\sigma'^{\vee} \cap \widetilde{M'}] \to A[\sigma^{\vee} \cap \widetilde{M}]$$

and corresponding morphism of R-schemes

$$\phi_* : \mathfrak{X}_\sigma \to \mathfrak{X}_{\sigma'}.$$

By gluing schemes and morphisms, we thus have a morphism of R-schemes $\mathfrak{X}(\Delta) \to \mathfrak{X}(\Delta')$. Compatibility with the toric structure is clear.

Proposition 2.1.16. *Notations as above, $\phi_* : \mathfrak{X}(\Delta) \to \mathfrak{X}(\Delta')$ is proper if and only if*

$$|\Delta| = \phi^{-1}(|\Delta'|).$$

Proof of this proposition will be postponed after we discuss lattice coarsening and base change (see 2.1.24).

Proposition 2.1.17. *$\mathfrak{X}_\sigma$ is a regular scheme if and only if $\sigma \cap \widetilde{N}$ can be generated by a subset of a basis of $\widetilde{N}$, hence $\mathfrak{X}(\Delta)$ is regular if and only if each cone $\sigma \in \Delta$ has such property.*

Proof. We may assume $\sigma \not\subset N_{\mathbb{Q}}$ otherwise $\mathfrak{X}_\sigma$ is an ordinary toric variety over K, and is proved, for example, in [5]. If $\sigma \cap \widetilde{N}$ is generated by part of a basis $e_1, \ldots, e_r$, complete it to a full basis of $\widetilde{N}$, $e_{r+1}, \ldots, e_{n+1}$ such that $e_i \in \widetilde{N}^+$. Let $e_1^*, \ldots, e_{n+1}^*$ be the dual basis, then $\sigma^\vee \cap \widetilde{M}$ is generated by $e_1^*, \ldots, e_r^*, \pm e_{r+1}^*, \ldots, \pm e_{n+1}^*$. Assume

$$e = a_1 e_1^* + \cdots + a_{n+1} e_{n+1}^*$$

where a_i are non-negative integers. Then the affine ring of $\mathfrak{X}_\sigma$ is isomorphic to

$$A[\sigma^\vee \cap \widetilde{M}] \simeq R[x_1, \ldots, x_r, x_{r+1}^{\pm}, \ldots, x_{n+1}^{\pm}]/(x_1^{a_1} \cdots x_{n+1}^{a_{n+1}} - t),$$

which is easily seen to be a regular ring.

Conversely, if $A[\sigma^\vee \cap \widetilde{M}]$ is a regular ring, we may assume σ is full dimensional, consider the maximal ideal $\mathfrak{m}$ generated by $\chi^m t^r$ for $\widetilde{m} = (m, r) \in \sigma^\vee \cap \widetilde{M}$ and $\widetilde{m} \neq 0$. By regularity,

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = n + 1.$$

Since $\sigma^\vee$ is full dimensional and strictly convex, the above equality implies that there are $n+1$ rays (1-dimensional faces) of $\sigma^\vee$, and any lattice points in

$\sigma^\vee \cap \widetilde{M}$ except 0 and the first lattice points of the rays can be written as a sum of two nonzero lattice ponts in $\sigma^\vee$, this implies that $\sigma^\vee$ is a strictly simplicial cone, and so is σ. $\square$

2.1.18. Let $\Delta(r)$ be the set of r-dimensional cones in Δ; for a ray $\rho \in \Delta(1)$, let v_ρ be the first lattice point on ρ. The orbit closure V_ρ is an invariant divisor by proposition 2.1.13. For any $\widetilde{m} = (m, r) \in \widetilde{M}$, $\chi^m t^r$ is a rational function on $\mathfrak{X}(\Delta)$.

Proposition 2.1.19. *The valuation of $\chi^m t^r$ along V_ρ is*

$$\mathrm{val}_{V_\rho} \chi^m t^r = \langle v_\rho, \widetilde{m} \rangle.$$

Proof. It suffices to consider the case when Δ is consisted of a single ray ρ. Let $e_1, e_2, \ldots, e_{n+1}$ be a basis of $\widetilde{N}$ such that $e_1 = v_\rho$, let $e_1^*, \ldots, e_{n+1}^*$ be the dual basis of $\widetilde{M}$. Then

$$\rho^\vee \cap \widetilde{M} = \mathbb{N}e_1^* \oplus \mathbb{Z}e_2^* \oplus \cdots \oplus \mathbb{Z}e_{n+1}^*.$$

If we let $X_i = \chi^{e_i^*}$, and write

$$e = \sum_{i=1}^{n+1} a_i e_i^*$$

with $a_i \in \mathbb{Z}$, and $a_1 \geq 0$, then the affine coordinate ring is

$$\mathcal{O}(\mathfrak{X}_\rho) = A[\rho^\vee \cap \widetilde{M}] \simeq R[x_1, x_2^\pm, \ldots, x_{n+1}^\pm]/(x_1^{a_1} \cdots x_{n+1}^{a_{n+1}} - t).$$

The generic point of the V_ρ corresponds to the prime ideal P generated by the monomials $\chi^{\widetilde{m}}$ with $\widetilde{m} \in \rho^\vee \cap \widetilde{M}$ and $\widetilde{m} \notin \rho^\perp$. Define a valuation w on $K(\mathfrak{X}_\rho) = K(M)$ to be

$$w(\sum_m a_m \chi^m) = \min\{\langle (m, v(a_m)), v_\rho \rangle\}$$

for $\sum a_m \chi^m \in K[M]$, a finite sum with $a_m \neq 0$. It is easy to check the valuation ring of w is exactly $\mathcal{O}_P$, thus the valuation of $\chi^m t^r$ is

$$\mathrm{val}_{V_\rho} \chi^m t^r = \langle v_\rho, \widetilde{m} \rangle.$$

$\square$

2.1.20. We are mostly interested in toric R-schemes with reduced special fiber. The irreducible components of the special fiber correspond to rays $\rho \in \Delta(1)$ which is not contained in $N_{\mathbb{Q}}$. By the above proposition, the multiplicity of V_ρ is $\mathrm{val}_{V_\rho} t = \langle e, v_\rho \rangle = p(v_\rho)$, where p is the projection map $\widetilde{N} \to \mathbb{Z}$. Thus we obtained the following.

Corollary 2.1.21. *The special fiber of $\mathfrak{X}(\Delta)$ is reduced if and only if for every ray $\rho \in \Delta(1)$ not contained in $N_{\mathbb{Q}}$, $p(v_\rho) = 1$.*

2.1.22. This can also be shown by the affine ring, let $\sigma \in \Delta$, then $\mathcal{O}(\mathfrak{X}_\sigma \times_R k) = k[\sigma^\vee \cap \widetilde{M}]/(\chi^e)$, the effect of modulo (χ^e) is to make $\chi^{\widetilde{m}} = 0$ if $\widetilde{m} - e \in \sigma^\vee \cap \widetilde{M}$. If $\mathrm{pr}_2(v_\rho) = 0$ or 1, for all $\rho < \sigma$, then $\chi^{\widetilde{m}} = 0$ for all $\widetilde{m}$ in the interior of $\sigma^\vee$, hence $k[\sigma^\vee \cap \widetilde{M}]/(\chi^e)$ is reduced.

2.1.23. If $\mathfrak{X}(\Delta)$ does not have reduced special fiber, we can make it reduced by a base change analogous to Mumford's semistable reduction [16]. In the toric scheme case, this is done by lattice coarsening. Let $\widetilde{N}$ be as usual with a projection $p : \widetilde{N} \to \mathbb{Z}$, denote $\widetilde{N}[d]$ to be $p^{-1}(d\mathbb{Z}) = N \oplus d\mathbb{Z}$ for a positive integer d. For an admissible fan Δ in $\widetilde{N}$, we write $\Delta[d]$ to mean the same fan but considered in $\widetilde{N}[d]$ instead.

Proposition 2.1.24. *For an admissible fan Δ in $\widetilde{N}$, there exists an integer $d > 0$ such that for any discrete valuation ring R', $\mathfrak{X}(\Delta[d])$ as a toric R'-scheme has reduced special fiber, and furthermore when $R' = R[t^{1/d}]$, $\mathfrak{X}(\Delta[d])$ is obtained from $\mathfrak{X}(\Delta)$ via base change $\operatorname{Spec} R[t^{1/d}] \to \operatorname{Spec} R$ followed by normalization.*

Proof. Let d be the least common multiple of v_ρ for all $\rho \not\subset N$, then $\Delta[d]$ satisfies the assumption in the corollary, hence $\mathfrak{X}(\Delta[d])$ has reduced special fiber. Now assume $R' = R[t^{1/d}]$, we show that $\mathfrak{X}(\Delta[d])$ is the normalization of $\mathfrak{X}(\Delta) \times_R R[t^{1/d}]$.

This is a local question, we show this for each $\sigma \in \Delta$. Note that the dual of $N \oplus d\mathbb{Z}$ is $M \oplus \frac{1}{d}\mathbb{Z}$ and we have a natural embedding $\widetilde{M} \subset M \oplus \frac{1}{d}\mathbb{Z}$ coresponding to $N \oplus d\mathbb{Z} \subset \widetilde{N}$. Let $\sigma[d]$ be the same cone but considered in $N \oplus d\mathbb{Z}$, and $S_{\sigma[d]}$ the monoid $\sigma^\vee \cap (M \oplus \frac{1}{d}\mathbb{Z})$.

The affine ring of $\mathfrak{X}_\sigma \times_R R[t^{1/d}]$ is $R[\chi^m t^s]_{(m,s)\in S_\sigma} \otimes_R R[t^{1/d}]$, the effect of this tensor product is just introducing $\frac{1}{d}e$ in S_σ i.e. if S' is the monoid generated by S_σ and $\frac{1}{d}e$ in $S_{\sigma[d]}$, then

$$R[\chi^m t^s]_{(m,s)\in S_\sigma} \otimes_R R[t^{1/d}] = R[\chi^m t^s]_{(m,s)\in S'}.$$

The effect of normalization is saturating S', which is $S_{\sigma[d]}$. $\square$

2.1.25. The above proposition shows that we can reduce the special fiber by a base extension. If $\mathfrak{X}(\Delta)$ already has reduced special fibre, then $\mathfrak{X}(\Delta[d]) = \mathfrak{X}(\Delta) \times_R R[t^{1/d}]$, this is because S' (the monoid generated by S_σ and $\frac{1}{d}e$ in $S_{\sigma[d]}$) is saturated, i.e. $S' = S_{\sigma[d]}$.

Proof of Proposition 2.1.16. First assume the map $\mathfrak{X}(\Delta) \to \mathfrak{X}(\Delta')$ is proper. Let $u \in \widetilde{N}$ be a primitive lattice point such that $p(u) = d > 0$ and $\phi(u) \in |\Delta'|$. From the map of cones below

$$\begin{tikzcd} & \widetilde{N}[d] & \Delta[d] \arrow[l] \arrow[dl, "\phi"'] \arrow[d, "p"] \\ \mathbb{N} \arrow[ur] \arrow[r] & \Delta'[d] \arrow[r, "p'"] & d\mathbb{N} \end{tikzcd}$$

where the map $\mathbb{N} \to \widetilde{N}[d]$ and $\mathbb{N} \to \Delta'[d]$ are given by $1 \mapsto u$ and $1 \mapsto \phi(u)$ respectively, we obtain a commutative diagram of morphisms of schemes

$$\begin{tikzcd} \operatorname{Spec} K(t^{1/d}) \arrow[r] \arrow[d] & \mathfrak{X}(\Delta[d]) \arrow[d] \\ \operatorname{Spec} R[t^{1/d}] \arrow[r] \arrow[ur, dashed] & \mathfrak{X}(\Delta'[d]) \end{tikzcd}$$

Since by assumption, $\mathfrak{X}(\Delta) \to \mathfrak{X}(\Delta')$ is proper, and $\mathfrak{X}(\Delta[d])$ (respectively

$\mathfrak{X}(\Delta'[d]))$ is obtained by base extension $\operatorname{Spec} R[t^{1/d}] \to \operatorname{Spec} R$ and normalization, hence $\mathfrak{X}(\Delta[d]) \to \mathfrak{X}(\Delta'[d])$ is also proper. Thus by the valuative criterion of properness, we can fill in the dotted arrow as in the above diagram. Assume $\operatorname{Spec} R[t^{1/d}]$ lands in some $\mathfrak{X}(\sigma[d])$ for some $\sigma \in \Delta$, this only possible if $u \in \sigma$. We conclude that $|\Delta| = \phi^{-1}(|\Delta'|)$.

Conversely, assume now $|\Delta| = \phi^{-1}(|\Delta'|)$, we shall use the valuative criterion of properness. In the valuative criterion, we can use only discrete valuation rings if it is a morphism $f : X \to Y$ of noetherian schemes of finite type ([10], II, Ex. 4.11). And if f is already separated, we can further assume that $\operatorname{Spec} K$ lands in a given open dense subset U of X in the valuative criterion. This can be shown by argument of contradiction, if f is not proper, then by Nagata's compactification theorem, we can compactify $X \subset \overline{X}$ over Y. Let $x \in \overline{X} \backslash X$ be a point, and $x_1 \in U$ such that x is a specialization of x_1. If Z is the closure of x_1, and K is the residue field of $\mathcal{O}_{Z,x_1}$, there is a discrete valuation ring R which dominates $\mathcal{O}_{Z,x}$, thus we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & \overline{X} \arrow[d] \\
\operatorname{Spec} R \arrow[r] \arrow[ur] & \overline{Y}
\end{tikzcd}$$

where $\operatorname{Spec} K \to \overline{X}$ is mapped to x_1 and $\operatorname{Spec} R \to \overline{X}$ is mapped to x_1 and x. This contradicts the valuative criterion, thus $f : X \to Y$ is proper.

Now in the following diagram, $\operatorname{Spec} F$ lands in the open torus $T_K \subset$

$\mathfrak{X}(\Delta)$,

$$\begin{tikzcd} \operatorname{Spec} K' \arrow[r] \arrow[d] & \mathfrak{X}(\Delta) \arrow[d] \\ \operatorname{Spec} R' \arrow[r] \arrow[ur, dashed] & \mathfrak{X}(\Delta'). \end{tikzcd}$$

Assume $\operatorname{Spec} R' \to \mathfrak{X}(\sigma')$ for some $\sigma' \in \Delta'$, this corresponds to a ring homomorphism

$$A[\sigma'^\vee \cap \widetilde{M}'] \to R'.$$

Thus we have a commutative diagram of ring homomorphisms

$$\begin{tikzcd} K' & K[M] \arrow[l] \\ R' \arrow[u] & A[\sigma'^\vee \cap \widetilde{M}'] \arrow[l] \arrow[u] \\ & R \arrow[ul] \arrow[u] \end{tikzcd}$$

Since the generic point of $\operatorname{Spec} R'$ lands on the generic fiber of $\mathfrak{X}(\Delta')$, the image of t in R' is nonzero. We see that there is an homorphism of abelian groups (need to choose a splitting)

$$\widetilde{M}' \to \widetilde{M} \to K'^* \to \mathbb{Z}$$

which takes non-negative values on $\sigma'^\vee \cap \widetilde{M}'$. The composition $\widetilde{M} \to \mathbb{Z}$ determines a lattice point $u \in \widetilde{N}$ which maps into σ'. Thus by assumption, there is a cone $\sigma \in \Delta$ which maps into σ' and contains u. This exactly determines the factorization

$$\operatorname{Spec} R' \to \mathfrak{X}(\sigma) \subset \mathfrak{X}(\Delta).$$

$\square$

2.1.26. We shall discuss the relative canonical divisor, this makes sense when $\mathfrak{X}(\Delta)$ has reduced special fiber. Now suppose $\mathfrak{X}(\Delta)$ is a toric R-scheme with reduced special fiber, let $\Delta(1) \subset \Delta$ be the subfan consisting of all 1-dimensional cones, then $\mathfrak{X}^0 = \mathfrak{X}(\Delta(1)) \subset \mathfrak{X}(\Delta)$ is an open T_R-invariant subscheme, and the complement of $\mathfrak{X}^0$ in $\mathfrak{X}(\Delta)$ is of codimension 2.

Proposition 2.1.27. *$\mathfrak{X}^0$ is smooth over $\operatorname{Spec} R$, let $K_{\mathfrak{X}^0}$ be the Cartier divisor class of the invertible relative canonical sheaf $\omega_{\mathfrak{X}^0/R} := \Omega^n_{\mathfrak{X}^0/R}$, then $K_{\mathfrak{X}^0} \sim -\Sigma_1 D_i \sim -\Sigma_2 D_j$ where Σ_1 runs over all T-invariant horizontal (i.e. not on the special fiber) prime divisors of $\mathfrak{X}^0$ and Σ_2 runs over all T-invariant prime divisors of $\mathfrak{X}^0$.*

Proof. Let $\rho \in \Delta(1)$, then $\mathfrak{X}(\rho)$ is isomorphic to $\mathbf{A}^1_K \times_K T^n_K$ if $\rho \subset N$, or T^n_R if $\rho \not\subset N$. In either case, it is smooth over $\operatorname{Spec} R$, thus $\mathfrak{X}^0$ is smooth over $\operatorname{Spec} R$.

Let $e_1, \ldots, e_n$ be a basis of M, and $x_i = \chi^{e_i}$. The rational n-form

$$\frac{\mathrm{d}\, x_1}{x_1} \wedge \frac{\mathrm{d}\, x_2}{x_2} \wedge \cdots \wedge \frac{\mathrm{d}\, x_n}{x_n}$$

differs by a factor of ± 1 if we choose a different basis of M. It is regular on T_K and on the special fiber. For any $\rho \in \Delta(1)$ with $\rho \subset N$, we can choose a basis of N one of which is the primitive lattice point on ρ. If we use the dual basis to write the n-form, clearly it has a pole of order 1 along V_ρ, thus $K_{\mathfrak{X}^0} \sim -\sum_1 D_i$. The regular function t on $\mathfrak{X}^0$ vanishes on the special fiber of order 1, so the sum of vertical T-invariant prime divisors is trivial, thus the result follows. $\square$

Corollary 2.1.28. *If $\mathfrak{X}(\Delta)$ has reduced special fiber, then $K_{\mathfrak{X}(\Delta)} + \Sigma_2 D_j$ is trivial*

Proof. Recall that $K_{\mathfrak{X}(\Delta)} = j_* K_{\mathfrak{X}^0}$ where j is the open immersion $K_{\mathfrak{X}^0} \subset K_{\mathfrak{X}}$, it is clear from the above proposition. $\square$

2.2 Construction from a polyhedron

2.2.1. Let V be a finite dimensional real vector space, a *convex polyhedron* $P \subset V$ is defined by the solution of a system of equalities and inequalies of linear forms. To be precise, choose a basis of V, let $x_1, \ldots, x_n$ be the coordinates on V, a polyhedron is given by the solution of some equalities or inequalities

$$a_1 x_1 + \cdots + a_n x_n = b,$$

or

$$a_1 x_1 + \cdots + a_n x_n \geq b.$$

A convex polyhedron is called *rational* if all coefficients a_i, b can be chosen as rational numbers. In our case, we need a rational convex polyhedron P in $\widetilde{M_{\mathbb{R}}} = M_{\mathbb{R}} \oplus \mathbb{R}$.

Definition 2.2.2. A rational convex polyhedron $P \subset \widetilde{M}_{\mathbb{R}}$ is called *admissible*, if

1. P is full dimensional.

2. vertices of P are integral.

3. projection of P to e-axis is bounded below.

2.2.3. Let $\overline{P}$ be the image of the projection of P in $M_{\mathbb{R}}$. Our definition is slightly different from [20]. In [20], the author requires that $\overline{P}$ is bounded, and for each lattice point $m \in \overline{P} \cap M$, the smallest real number a such that $m + (0, a) \in P$ is integral. The first condition will imply that the toric scheme constructed is proper over $\operatorname{Spec} R$, the second condition is equivalent to that the special fiber being reduced as we will see later.

2.2.4. To construct a toric R-scheme from an admissible polyhedron $P \subset \widetilde{M}_{\mathbb{R}}$, we put P in space $\widetilde{M}_{\mathbb{R}} \oplus \mathbb{R}$ with the extra coordinate 1. Let $C(P)$ be the closure of the cone over P, i.e.

$$C(P) = \{(ap, a) | p \in P, a \geq 0\}^{-}.$$

Define A_P to be $A[C(P) \cap (\widetilde{M} \oplus \mathbb{Z})]$ as in 2.1.3, i.e.

$$A_P = k[C(P) \cap (\widetilde{M} \oplus \mathbb{Z})] \otimes_{k[t]} R,$$

where the map $k[t] \to k[C(P) \cap (\widetilde{M} \oplus \mathbb{Z})]$ is given by $t \mapsto \chi^e$. Note that

$C(P) \cap (\widetilde{M} \oplus \mathbb{Z})$ is a finitely generated monoid, A_P is a finitely generated R-algebra, it is graded by the extra coordinate.

Definition 2.2.5. Define $\mathfrak{X}_P$ to be $\operatorname{Proj} A_P$.

2.2.6. To relate $\mathfrak{X}_P$ with toric R-schemes constructed from relative fans, we can recover the fan from P. Recall that the normal fan of P, denoted by Δ_P, is constructed as follows. For each face $F < P$, let $x \in F$ be a relative interior point, let $\sigma_F^\vee$ be the cone in $\widetilde{M}_{\mathbb{R}}$ spanned by all vectors $v - x$ for $v \in P$. It is easy to see that $\sigma_F^\vee$ is independent of the choice of x.

Let $\sigma_F \subset \widetilde{N}_{\mathbb{R}}$ be the dual cone of $\sigma_F^\vee$, and Δ_P be the collection of all σ_F for F running over all faces of P

Proposition 2.2.7. *If P is an admissible polyhedron, then Δ_P is an admissible fan.*

Proof. The fact that Δ_P is a fan is a general result, which is true for any convex polyhedron. If G is a face of F, let $g \in G$, $f \in F$ be relative interior points respectively. For any $v \in P$, $\epsilon(v - g) + f \in P$ for $\epsilon > 0$ sufficiently small, thus $\sigma_G^\vee \subset \sigma_F^\vee$ and $\sigma_G \supset \sigma_F$. Moreover since $\mathbb{R}(f - g) \subset \sigma_F^\vee$, we see that σ_F is contained in $(f - g)^\perp$, and σ_G is contained in the half space $\langle f - g, \cdot \rangle \geq 0$, thus $\sigma_F = \sigma_G \cap (f - g)^\perp$ is a face of σ_G.

For any two faces F and G, let H be the smallest face of P that contains F and G as its faces. σ_H is a common face of σ_F and σ_G, We conclude that Δ_P is a fan.

Since P is admissible, e is contained in every $\sigma_F^\vee$, thus Δ_P is also admissible. □

2.2.8. From the proof, it is clear that there is a one-to-one correspondence between the faces of P and the cones of Δ_P in the reversing order. Moreover, the cones in Δ_P which is contained in $N_{\mathbb{R}}$ corresponds to faces F of P which satisfy $F + \mathbb{R}_{\geq 0} = F$.

Proposition 2.2.9. $\mathfrak{X}_P = \mathfrak{X}(\Delta_P)$.

Proof. Let F be a vertex of P, identify χ^F with the homogeneous element of A_P of degree 1, then

$$\mathrm{Spec}(A_P)_{((\chi^F))} \subset \mathfrak{X}_P$$

is an open affine subvariety.

It is clear that

$$(A_P)_{((\chi^F))} = A[\sigma_F^\vee \cap \widetilde{M}].$$

Thus $\mathrm{Spec}(A_P)_{((\chi^F))}$ is identified with $\mathfrak{X}(\sigma_F)$. $\mathfrak{X}(\Delta_P)$ is covered by all $\mathfrak{X}(\Delta_F)$ where F runs over all vertices of P, $\mathfrak{X}_P$ is also covered by all $\mathrm{Spec}(A_P)_{((\chi^F))}$ where F runs over all vertices of P, this is because A_P is generated over the degree 0 part by χ^F (possibly after saturation), i.e. for any $m \in C(P)$ of degree > 0, we can write

$$am = \sum_F b_F F + m'$$

for some $a > 0, b_F \geq 0$ and $m' \in C(P) \cap (\widetilde{M} \oplus 0)$. In the summation, F runs over all vertices of P.

The gluing is also compatible. Given two vertices $F, G < P$, let H be the smallest face of P that contains F and G. We claim that

$$(\operatorname{Spec}(A_P)_{((\chi^F))}) \cap (\operatorname{Spec}(A_P)_{((\chi^G))}) = \operatorname{Spec}(A_P)_{((\chi^F\chi^G))} = \operatorname{Spec} A[\sigma_H^\vee].$$

Indeed, it is equivelent to the fact that $\sigma_F^\vee + \sigma_G^\vee = \sigma_H^\vee$. $\square$

Immediately from the above result and results in previous section, we have following corollaries.

Corollary 2.2.10. *$\mathfrak{X}_P$ is proper over* $\operatorname{Spec} R$ *if and only of $\overline{P}$, the image of P in the projection to $M_{\mathbb{R}}$ is bounded.*

Proof. $\mathfrak{X}_P$ is proper if and only if $|\Delta_P| = \widetilde{N}_{\mathbb{R}}^+$. We show that the latter condition is equivalent to $\overline{P}$ being bounded.

If $\overline{P}$ is bounded, then every unbounded face F of P has the property that

$$F + \mathbb{R}_{\geq 0} \cdot e = F.$$

These faces correspond to the faces of $\overline{P}$, and so do their dual cones. Thus the union of σ_F where F runs over all unbounded faces of P is $N_{\mathbb{R}}$. Given $v \in \widetilde{N}_{\mathbb{R}}$ with $p(v) > 0$, the face of P which minimize the linear fucntion $\langle v, \cdot \rangle$ is a bounded face G, then $v \in \sigma_G$. We see that the support of Δ_P is $\widetilde{N}_{\mathbb{R}}^+$.

Conversely if $\overline{P}$ is not bounded, we can find an unbounded facet $F < P$, such that the inner normal vection $v \in \widetilde{N}$ has $p(v) > 0$. Assume F is defined by

$$\langle v, \cdot \rangle = a.$$

We can perturb v a little bit, i.e. $v' \in \widetilde{N}_{\mathbb{R}}$ is close to v, such that $p(v') > 0$ and the equation

$$\langle v', \cdot \rangle = a$$

intersects F in the interior. There is no face of P which can minimize the fuction $\langle v, \cdot \rangle$, thus v' is not in the support of Δ_P.

$\square$

Corollary 2.2.11. *The special fiber of $\mathfrak{X}_P$ is reduced if and only if the first lattice point above any $m \in \overline{P} \cap M$ is on the lower boundary of P.*

Proof. This is an easy consequence of 2.1.21. We know that the special fiber of $\mathfrak{X}_P$ is reduced if and only if $p(v_\rho) = 1$ for every $\rho \in \Delta_P[1]$, where v_ρ is the first lattice point on ρ. This property is equivalent to the property of P that for each $m \in \overline{P} \cap M$, the smallest number $a \in \mathbb{R}$ such that $(m, a) \in P$ is integral.

Indeed if $p(v_\rho) = 1$, let F be the corresponding facet of P, then F is defined by equation

$$\langle v_\rho, \cdot \rangle = r.$$

Since the vertices of F is integral, r is also integral. For any $m \in \overline{P} \cap M$,

$$\langle v_\rho, (m, a) \rangle = \langle v_\rho, (m, 0) \rangle + a.$$

Thus the verticle line $m + \mathbb{R} \cdot e$ hits the polyhedron P at

$$a = r - \langle v_{rho}, (m, 0) \rangle,$$

which is integral.

Conversely if condition (5) is satisfied. Let F be a facet of P which is not verticle (i.e. $F + \mathbb{R}_{\geq 0} \cdot e \neq F$) and let v be its primitive inner normal vector, clearly $p(v) > 0$. Suppose F is defined by the equation

$$\langle v, \cdot \rangle = r.$$

r is certainly an integer. Let $\overline{F}$ be the image of F under the projection to $M_{\mathbb{R}}$, then $\overline{F}$ is of full dimension and with integral vertices. Thus we can find $m_1, \ldots, m_{n+1} \in \overline{F} \cap M$ ($n = \dim M_{\mathbb{R}}$) which are the vertices of a primitive simplex (i.e. the volume of the convex hull is $1/n!$). Since the vectors $m_i - m_{n+1}$ generates the full lattice M, this in turn implies that for all $m \in M$, the line $m + \mathbb{R} \cdot e$ hits the hyperplane

$$\langle v, \cdot \rangle = r$$

at a lattice point. Since v is primitive, we can find $u \in \widetilde{M}$ with $\langle v, u \rangle = 1$. Therefore $\langle v, (r-1)u \rangle = r - 1$, and it is only possible that $\langle v, (r-1)u + e \rangle = r$, hence $p(v) = 1$. $\square$

2.3 Construction from orbit closure

2.3.1. Our third construction is simply by taking oribt closure in $\mathbb{A}^n_R$ or $\mathbb{P}^n_R$. Let $T \subset T^N$ be a subtorus, for any K-point x of T^N, we may take closure of the T-orbit of x in $\mathbb{A}^N_R$ or $\mathbb{P}^N_R$.

2.3.2. First consider the affine case. A subtorus $T \subset T^N$ is equivalent to a surjective map $M_{T^N} \to M_T$. If we choose a basis of M_{T^N} (compatible with $\mathbb{A}^N$) and identify M_T with $\mathbb{Z}^n$, this is equivalent to give N elements $m_1, \ldots, m_N$ of M_Y which generate (as group) M_Y. Assume the coordinate of x under this basis is $x = (a_1, \ldots, a_N)$, since $x \in T^N(K)$, all a_i are nonzero. Let v_i be the valuation of a_i, let C be the cone in $\widetilde{M}_{\mathbb{R}} = (M_Y \oplus \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ generated by all (m_i, v_i) and $e = (0, 1)$.

Proposition 2.3.3. *The normalization of the closure of $T_K \cdot x$ in $\mathbb{A}^N_R$ is isomorphic to* $\operatorname{Spec} A[C \cap \widetilde{M}]$.

Proof. let $x_1, \ldots, x_N$ be the coordinates of $\mathbb{A}^N$, then $T_K \subset \mathbb{A}^N_K$ is given by

$$\operatorname{Spec} K[x_1^{\pm}, \ldots, x_N^{\pm}] \subset \operatorname{Spec} K[x_1, \ldots, x_N].$$

The closed immersion $T_K \subset T^N_K$ is given by the surjection of rings

$$K[x_1^{\pm}, \ldots, x_N^{\pm}] \to K[M_Y]$$

defined as $x_i \mapsto \chi^{m_i}$. The orbit $T_K \cdot x$ in T^N_K is given by surjection of rings

$$K[x_1^{\pm}, \ldots, x_N^{\pm}] \to K[M_Y]$$

defined by $x_i \mapsto \chi^{m_i} a_i$. Taking closure is equivalent to taking the image of the same map from $R[x_1, \ldots, x_n]$ to $K[M_Y]$, thus the coordinate ring of the image is isomorphic to $A[C \cap \widetilde{M}]$ □

As a corollary of the above result, we have the following.

Corollary 2.3.4. *The closure of $T_K \cdot x$ is normal if and only if the monoid generated by (m_i, v_i) and e in $\widetilde{M}$ is saturated (in $\widetilde{M}$).*

2.3.5. We now consider taking closure in $\mathbb{P}^N_R$, again $T_K \subset T^N_K$ is a subtorus. Let $x_0, \ldots, x_N$ be the homogeneous coordinates of $\mathbb{P}^N$, $x = (a_0, \ldots, a_N) \in \mathbb{P}^N(K)$ is a K-point on the generic fiber with non-zero coordinates. The characters of T^N may be identified with $(\oplus \mathbb{Z} \cdot e_i)/\mathbb{Z} \cdot (e_0 + \cdots + e_N) = M_{T^N}$, by abuse of notation, we still use e_i to mean the image of them in the quotient space.

Definition 2.3.6. Let $\widetilde{M} = M \oplus \mathbb{Z}$ be a lattice with a distinguised e-axis. Given a finite set of lattice points $\mathcal{A} = p_1, \ldots, p_r$, the *admissible convex hull* $P(\mathcal{A})$ of $\mathcal{A}$ is defined as translation in the positive e-direction of the convex hull of $\mathcal{A}$. To be precise, if $\mathrm{conv}(\mathcal{A})$ is the convex hull of $\mathcal{A}$, then

$$P(\mathcal{A}) = \{p + \lambda e | p \in \mathrm{conv}(\mathcal{A}), \lambda \geq 0\}.$$

2.3.7. Clearly if $P(\mathcal{A})$ is of full dimension, then it is an admissible convex polyhedron. We now describe the closure of $T \cdot x$ in $\mathbb{P}^N_R$. The closed immersion of $T \subset T^N$ is given by a surjective morphism of lattices $M_{T^N} \to M_T$. Let m_i be the image of e_i. If v_i is the valuation of a_i, let P be the admissible convex hull of the set

$$\{(m_i, v_i) | i = 0, \ldots, N\} \subset \widetilde{M} = M_T \oplus \mathbb{Z},$$

we have the following result.

Proposition 2.3.8. *The normalization of the closure of $T_K \cdot x$ in $\mathbb{P}^N_R$ is isomorphic to the toric R-scheme $\mathfrak{X}_P$.*

Proof. Let $\mathfrak{X}$ be the closure of $T_K \cdot x$ in $\mathbb{P}^N_R$, and

$$\pi : \widetilde{\mathfrak{X}} \to \mathfrak{X}$$

be the normalization in the function field of $\mathfrak{X}$. If $x_0, \ldots, x_N$ are the homogeneous coordinates of $\mathbb{P}^N$, let $\mathfrak{X}_i := \mathfrak{X} \cap (x_i \neq 0)$ be the affine open subset of $\mathfrak{X}$, then we may think $\mathfrak{X}_i$ is an orbit closure in $\mathbb{A}^N_R$.

The affine coordinates of x is given by

$$(\frac{a_0}{a_i}, \ldots, \frac{a_N}{a_i}),$$

with valuations

$$(v_0 - v_i, \ldots, v_N - v_i)$$

, and the lattice map $M_{Y^N} \to M_Y$ is given by $m_0 - m_i, \ldots, m_N - m_i$. Let C_i be the cone in $\widetilde{M_{\mathbb{R}}}$ generated by vectors

$$(m_0, v_0) - (m_i, v_i), \ldots, (m_N, v_N) - (m_i, v_i),$$

then by proposition ??, $\pi^{-1}(X_i)$ is given by $\operatorname{Spec} A[C_i \cap M]$. Thus $\widetilde{(}\mathfrak{X})$ is constructed by gluing $\operatorname{Spec} A[C_i \cap M]$, which is exactly the toric R-scheme associated with the polyhedron P. $\square$

2.3.9. When is $\mathfrak{X}$ normal? Of course $\mathfrak{X}$ is normal if and only if each affine piece $\mathfrak{X}_i$ is normal, this is equivalent to say that for each (m_i, v_i), if it is a vertex of P, then $C_i \cap \widetilde{M}$ is generated by vectors

$$(m_0, v_0) - (m_i, v_i), \ldots, (m_N, v_N) - (m_i, v_i).$$

We may also ask when is $\mathfrak{X}$ projectively normal. It is projectively normal if A_P is integrally closed, i.e. $C(P) \cap \widetilde{M} \oplus \mathbb{Z}$ is saturated. This is equivalent to following: for any lattice point $\widetilde{m} \in C(P) \cap \widetilde{M}$ of degree r, $\widetilde{m}$ can be written as a sum of r lattice points in $C(P) \cap \widetilde{M}$ of degree 1 and some multiple of e.

Chapter 3

Tropical Compactification in Non-constant Coefficient Case

3.1 Definition and basic properties

We first define tropical and schön compactification for $Y \subset T$ over a field with discrete valuation and establish parallel results as in the constant coefficient case. Notations as in the previous section, let $Y \subset T_K$ be a subvariety, $\mathrm{trop}(Y)$ in this case is taken to be the tropicalization of $Y_{\mathbb{K}} \subset T_{\mathbb{K}}$.

Note that $\mathrm{trop}(Y)$ sits inside $N_{\mathbb{Q}}$, define $\mathcal{T}(Y)$ to be the closure of the set $\{(t \cdot x, t) \in N_{\mathbb{Q}} \oplus \mathbb{Q} = \widetilde{N}_{\mathbb{Q}} | x \in \mathrm{trop}(Y), t \in \mathbb{Q}_{>0}\}$. We take $\mathcal{T}(Y)$ as a replacement of $\mathrm{trop}(Y)$ in the non-constant coefficient case. If we modify the BG-set definition of $\mathrm{trop}(Y)$ (definition 1.1.3, (2)), $\mathcal{T}(Y)$ coincides with the following set:

$$\{(u|_{M_{\mathbb{Q}}}, u(t)) \in N_{\mathbb{Q}} \oplus \mathbb{Q} | u : \mathcal{O}(Y) \to \mathbb{Q}\}$$

where u runs over all valuations trivial on R^* and non-negative on R. Let $\Delta \subset \widetilde{N}_{\mathbb{Q}}$ be an admissible fan, $\mathfrak{X}(\Delta)$ the corresponding toric scheme over R, $\overline{Y}$ the closure of Y in $\mathfrak{X}(\Delta)$, we make the following definitions.

Definition 3.1.1. We say $\overline{Y}$ is a *tropical compactification* or Δ is a *tropical fan* if $\overline{Y}$ is proper over R and the structure map $T_R \times_R \overline{Y} \to \mathfrak{X}(\Delta)$ is flat and surjective.

Definition 3.1.2. $\overline{Y}$ is called a *schön compactification* if it is tropical and moreover the structure map is smooth. We say Y is *schön* in T if it admits a schön compactification, and we say Y is *schön* if it is schön in the intrinsic torus.

Definition 3.1.3. $\overline{Y}$ is called a *hübsch compactification* if it is schön, $\mathfrak{X}(\Delta)$ has reduced special fibre and $K_{\overline{Y}} + B_{\overline{Y}}$ is ample. We say Y is *hübsch* in T if it admits a hübsch compactification, and we say Y is *hübsch* if it is hübsch in the intrinsic torus.

Remark 3.1.4. Since $\mathfrak{X}(\Delta)$ has reduced fiber and $\overline{Y}$ has smooth structure map, $K_{\mathfrak{X}}$ and $K_{\overline{Y}}$ are well defined. We have $K_{\mathfrak{X}}+B_{\mathfrak{X}} = 0$, and by adjunction formula, $K_{\overline{Y}} + B_{\overline{Y}} = \det \mathscr{N}_{\overline{Y}/\mathfrak{X}(\Delta)}$.

3.1.5. Here we give a geometric description of $\mathcal{T}(Y)$ analogous to 1.1.14. Let Y be a normal very affine variety over K, suppose we have a compactification $Y \subset \overline{Y}$ with $\overline{Y}$ a normal scheme over $\operatorname{Spec} R$ and simple normal corssing divisorial boundary. For each boundary divisor E, it induces a valuation

$$\mathrm{val}_E : \mathcal{O}(Y)^* \to \mathbb{Z},$$

which is trivial on R^* and nonnegative on R. Hence we may think val_E is a lattice in $\widetilde{N}$, and in fact it is in $\widetilde{N}^+$. For a collection $\sigma = \{E_i : i \in I\}$ of boundary divisors with $\cap E_i \neq \emptyset$, let F_σ be a cone in $\widetilde{N}_{\mathbb{Q}}$ generated by all val_{E_i} for $E_i \in I$. We have the following result.

Proposition 3.1.6. $\mathcal{T}(Y) = \cup_\sigma F_\sigma$ *where* σ *runs over all collections of boundary divisors with nonempty intersection?*

Proof. First we show that $\mathcal{T}(Y) \supset \cup_\sigma F_\sigma$. This follows from the Bieri-Grove set description of $\mathcal{T}(Y)$. Clearly $\mathrm{val}_E \in \mathcal{T}(Y)$ for every boundary divisor E. For a collection of boundary divisors $\sigma = \{E_i : i \in I\}$, any $\sum_{i \in I} a_i \,\mathrm{val}_{E_i}$ with $a_i \in \mathbb{Z}_{\geq 0}$ is a valuation induced by the exceptional divisor of the weighted blow-up. Thus $F_\sigma \subset \mathcal{T}(Y)$.

Next we prove the converse. Let u be a valuation $\mathcal{O}(Z)\backslash\{0\} \to \mathbb{Q}$ which is trivial on R^* and nonnegative on R for a subvariety $Z \subset Y$. Let $R(Z)$ be the valuation ring of u and $K(Z)$ the function field of Z, we have $R \subset R(Z)$. Since $\overline{Y}$ is proper, by the valuative criterion of properness, we have a morphism $\mathrm{Spec}\, R(Z) \to \overline{Y}$. Let σ be the collection of boundary divisors that contains the closed point of $\mathrm{Spec}\, R(Z)$, we then have $u \in F_\sigma$. Indeed let $\overline{Y}' := \overline{Y} \backslash \cup E_j$ where E_j runs over all boundary divisors not in σ, we have a morphism $\mathrm{Spec}\, R(Z) \to \overline{Y}'$ and consequently a map $\mathcal{O}(\overline{Y}') \to R(Z)$. For any lattice point $f \in F_\sigma^\vee \subset \widetilde{M}$, consider as a regular function on Y (up to a scalar in R^*) extends to a regular function on $\overline{Y}'$, thus $u(f) \geq 0$ and $u \in F_\sigma$. $\square$

Remark 3.1.7. The assumption in the above proposition can be weakened, it remains true if $Y \subset \overline{Y}$ is only a toroidal embedding. This follows as in the constant coefficient case that the conical complex that Mumford constructed will map onto $\mathcal{T}(Y)$, and this works in the non-constant coefficient case [16].

We now prove some basic properties of tropical compactification in the non-constant coefficient case.

Proposition 3.1.8. *$\overline{Y}$ is proper over R iff $|\Delta| \supset \mathcal{T}(Y)$. If Δ is a tropical fan, then $|\Delta| = \mathcal{T}(Y)$.*

Proof. This can be proved following the same idea as in the constant coefficient case in [24], here we proceed with a new proof using the BG-set definition and valuative criterion.

Suppose $\overline{Y}$ is proper over R, let u be a valuation $K(Z)^\times \to \mathbb{Q}$ which is trivial on R^* and nonnegative on R for some subvariety Z of Y. Let $R(Z)$ be the valuation ring, then we have a following commutative diagram,

$$\begin{tikzcd}
\operatorname{Spec} K(Z) \arrow[r] \arrow[dd] & \overline{Y} \arrow[r] & \mathfrak{X}(\Delta) \arrow[dd] \\
 & \mathfrak{X}_\sigma \arrow[ur] & \\
\operatorname{Spec} R(Z) \arrow[uur, "f"] \arrow[ur, "g"'] \arrow[rr] & & \operatorname{Spec} R\,,
\end{tikzcd}$$

where the existence of f is due to the valuative criterion of properness of $\overline{Y}$, f then factors through some open affine toric scheme $\mathfrak{X}_\sigma \subset \mathfrak{X}(\Delta)$. g corresponds to a ring homomorphism $\mathcal{O}(\mathfrak{X}_\sigma) \to R(Z)$, which implies that u takes non-negative values on S_σ, thus $[u] \in \sigma$ where $[u]$ denotes the image of u in $\widetilde{N}_{\mathbb{Q}}$. We proved $|\Delta| \supset \mathcal{T}(Y)$.

Suppose $|\Delta| \supset \mathcal{T}(Y)$, if $\overline{Y}$ is not proper, there is a proper R-scheme Y', containing $\overline{Y}$ as an open dense subscheme. Let y_1 be the generic point of Y' and y_0 any point of Y' not contained in $\overline{Y}$, then $\mathcal{O}_{Y',y_0}$ is dominated by a discrete valuation ring of $K(Y') = K(Y)$. Let u denote the valuation, and $R(Y)$ the valuation ring, then $[u] \in \sigma$ for some $\sigma \in \Delta$. The map $\operatorname{Spec} R(Y) \to Y'$ sending the generic point to y_1 and the closed point to y_0 factors through $\mathfrak{X}_\sigma$, a contradiction.

Let Δ be a tropical fan, we already proved $\Delta \supset \mathcal{T}(Y)$, hence for any Δ' refining Δ, $\overline{Y}(\Delta')$ is proper over R. Let $\mathfrak{Y} = \overline{Y}(\Delta) \times_{\mathfrak{X}(\Delta)} \mathfrak{X}(\Delta')$. We show that $\mathfrak{Y} = \overline{Y}(\Delta')$. Indeed we have a fibre diagram

$$\begin{tikzcd} T_R \times_R \mathfrak{Y} \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X}(\Delta') \arrow[d] \\ T_R \times_R \overline{Y}(\Delta) \arrow[r] & \mathfrak{X}(\Delta) . \end{tikzcd}$$

Since $\overline{Y}(\Delta)$ is tropical, the top arrow is also flat and surjective. Restricting on the open subscheme $T_K \subset \mathfrak{X}(\Delta')$, $f^{-1}(T_K) = T_K \times_K Y$ which is integral. By the lemma below, $T_R \times_R \mathfrak{Y}$ is integral, so is $\mathfrak{Y}$. Hence $\mathfrak{Y}$ is $\pi^{-1}(\overline{Y}(\Delta)) \subset \mathfrak{X}(\Delta')$ with reduced induced structure, hence $\overline{Y}(\Delta') = \mathfrak{Y}$, it is tropical and is the pullback of $\overline{Y}(\Delta)$.

It remains to prove that Δ is supported on $\mathcal{T}(Y)$. Suppose it is not, let Δ' refines Δ such that there is a subfan $\Delta'' \subset \Delta'$ with $|\Delta''| = \mathcal{T}(Y)$. $\mathfrak{X}(\Delta'')$ is a open subscheme of $\mathfrak{X}(\Delta')$, but not equal to $\mathfrak{X}(\Delta')$. $\overline{Y}(\Delta'') = \overline{Y}(\Delta')$, thus the structure map $T_R \times_R \overline{Y}(\Delta') \to \mathfrak{X}(\Delta')$ fails to be surjective. $\square$

Lemma 3.1.9. *Let $f : X \to Y$ be a flat morphism of schemes. Assume Y is integral. If there exists a dense Zariski open subset $U \subset Y$ such that $f^{-1}(U)$ is integral, then X is integral.*

Proof. For any $x \in X$, let $y = f(x)$. Take affine open neighbourhoods $\operatorname{Spec} B$ and $\operatorname{Spec} A$ of x and y respectively such that $f : \operatorname{Spec} B \to \operatorname{Spec} A$, then $A \to B$ is flat. We can find $a \in A$ such that $\operatorname{Spec} A_a \subset U$. Since $f^{-1}(\operatorname{Spec} A_a) \cap \operatorname{Spec} B = \operatorname{Spec} B_a$, we know B_a is a domain. Tensoring $0 \to A \to A_a$ with B over A, we have $0 \to B \to B_a$, hence B is a domain. X is reduced.

If X is not irreducible, there is an irreducible component X' of X such that $f : X' \to Y - U$. However each irreducible component of X should dominate Y by the openness of a flat map, this is a contradiction, which proves the lemma. □

Proposition 3.1.10. *Any refinement of a tropical fan is tropical, if $Y \subset T_K$ admits a schön compactification, then any tropical fan produces a schön compactification. Let $\overline{Y} \subset \mathfrak{X}(\Delta) = \mathfrak{X}$ be a schön compactification, then $\overline{Y}$ is locally a complete intersection in $\mathfrak{X}(\Delta)$. Furthermore for any refinement Δ' of Δ, let $\overline{Y}'$ be the closure of Y in $\mathfrak{X}(\Delta') = \mathfrak{X}'$ with proper birational map $\pi : \overline{Y}' \to \overline{Y}$, then $\det \mathscr{N}_{\overline{Y}'/\mathfrak{X}'} = \pi^*(\det \mathscr{N}_{\overline{Y}/\mathfrak{X}})$.*

Proof. Using the diagram in proposition 3.1.8, we have a fiber diagram

$$\begin{tikzcd} T_R \times_R \overline{Y}(\Delta') \arrow[r] \arrow[d] & \mathfrak{X}(\Delta') \arrow[d] \\ T_R \times_R \overline{Y}(\Delta) \arrow[r] & \mathfrak{X}(\Delta). \end{tikzcd}$$

It follows that if Δ is a schön fan, then so is Δ'. Since $T_R \times_R \mathfrak{X}(\Delta) \to \mathfrak{X}(\Delta)$ is smooth, then the closed embedding $T_R \times_R \overline{Y} \to T_R \times_R \mathfrak{X}(\Delta)$ is a locally complete intersection. By the fiber diagram above, it is clear that

$$\det \mathscr{N}_{\overline{Y}'/\mathfrak{X}'} = \pi^*(\det \mathscr{N}_{\overline{Y}/\mathfrak{X}}).$$

$\square$

Proposition 3.1.11. *Tropical fan exists assuming $Y \subset T$ is rigid.*

Before proving the existence of tropical compactifications, we need some preliminary results. Let $\mathbf{P}_R$ be $\mathbf{P}_R^n$ and P a numerical polynomial, we have the Hilbert scheme $\mathrm{Hilb}_P(\mathbf{P}_R/R)$, parametrizing all subschemes of $\mathbf{P}_R$ which are flat and proper over $\operatorname{Spec} R$ with Hilbert polynomial P. The generic fiber of $\mathrm{Hilb}_P(\mathbf{P}_R/R)$ is $\mathrm{Hilb}_P(\mathbf{P}_K/K)$ and the special fiber is $\mathrm{Hilb}_P(\mathbf{P}_k/k)$. Since $T_R = T_R^n$ acts on $\mathbf{P}_R$, T_R acts on $\mathrm{Hilb}_P(\mathbf{P}_R/R)$. This action is equivariant with respect to some embedding of $\mathrm{Hilb}_P(\mathbf{P}_R/R) \hookrightarrow \mathbf{P}_R^N$, i.e. this action is induced by a group scheme morphism $T_R^n \to T_R^N$. Thus the orbit closure of T_K at any K-point of $\mathrm{Hilb}_P(\mathbf{P}_R/R)$ is a (possibly non-normal) toric R-scheme.

Let $s : \operatorname{Spec} R \to \mathbf{P}_R$ be a section, then the (contravariant) functor $\mathrm{Hilb}_{P,s} : \mathbf{Sch}/R \to \mathbf{Set}$ defined by

$$\mathrm{Hilb}_{P,s}(Z) = \left\{ \begin{array}{l} \text{subschemes } V \subset Z \times_R \mathbf{P}_R \text{ which are flat and proper} \\ \text{over } Z \text{ with Hilbert polynomial } P \text{ such that the map} \\ Z \to Z \times_R \mathbf{P}_R \text{ by } (1_Z \times s) \text{ factors } Z \to V \subset Z \times_R \mathbf{P}_R \end{array} \right\}$$

is represented by $\mathrm{Hilb}_{P,s}(\mathbf{P}_R/R)$, a closed subscheme of $\mathrm{Hilb}_P(\mathbf{P}_R/R)$.

Indeed, let $\mathrm{Univ} \subset \mathrm{Hilb}_P(\mathbf{P}_R/R) \times_R \mathbf{P}_R$ be the universal family, identify $\mathrm{Hilb}_P(\mathbf{P}_R/R)$ with the closed subscheme of $\mathrm{Hilb}_P(\mathbf{P}_R/R) \times_R \mathbf{P}_R$ via $(\mathrm{Id} \times s)$, let $\mathrm{Hilb}_{P,s}(\mathbf{P}_R/R)$ be the scheme-theoretic intersection of Univ and $\mathrm{Hilb}_P(\mathbf{P}_R/R)$ in $\mathrm{Hilb}_P(\mathbf{P}_R/R) \times_R \mathbf{P}_R$, i.e. we have a following cartesian diagram

$$\begin{tikzcd}
\mathrm{Hilb}_{P,s}(\mathbf{P}_R/R) \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Hilb}_P(\mathbf{P}_R/R) \arrow[d, "(\mathrm{Id} \times s)"] \\
\mathrm{Univ} \arrow[r] & \mathrm{Hilb}_P(\mathbf{P}_R/R) \times_R \mathbf{P}_R\,.
\end{tikzcd}$$

It is clear that $(\mathrm{Hilb}_{P,s}(\mathbf{P}_R/R), i^* \mathrm{Univ})$ represents the functor $\mathrm{Hilb}_{P,s}$.

When s is $e : \operatorname{Spec} R \to T_R$, the identity of T_R, we call $\mathrm{Hilb}_{P,e}(\mathbf{P}_R/R)$ the *visible contour* of $\mathrm{Hilb}_P(\mathbf{P}_R/R)$. This generalizes Kapranov's visible contour of Grassmannian [14].

3.1.12. Next we recall Lafforgue transversality [17]. Let S be a noetherian scheme, X an S-scheme with an action of a group S-scheme G. Let $V \subset G \times_S X$ be a G-invariant closed subscheme, $X_{e,V}$ the scheme-theoretic intersection of V and $\{e\} \times X$ in $G \times_S X$, i.e. a cartesian diagram as follows,

$$\begin{tikzcd}
X_{e,V} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "(e \times 1_X)"] \\
V \arrow[r] & G \times_S X\,.
\end{tikzcd}$$

$X_{e,V}$ (or rather its image in X) is a closed subscheme of X.

Lemma 3.1.13. *The multiplication map $\varphi : G \times_S X_{e,V} \to V$ is an isomor-*

phism, and identifies the multiplication map $G \times_S X_{e,V} \to X$ with the second projection map $V \to X$.

For any G-equivariant S-morphism $\pi : \mathfrak{X} \to X$, $V' = \pi^*(V)$, then $\mathfrak{X}_{e,V'} = \pi^*(X_{e,V})$.

Proof. Let $\sigma : G \times_S X \to X$ denote the action, $\iota : G \to G$ the inverse. Let σ^{-1} denote the twisted action, i.e.

$$\sigma^{-1} : G \times_S X \xrightarrow{\iota \times 1_X} G \times_S X \xrightarrow{\sigma} X\,.$$

Note that

$$V \xrightarrow{(\mathrm{pr}_1, \sigma^{-1})} G \times_S X \xrightarrow{1_G \times (e, 1_X)} G \times_S (G \times_S X)$$

factors through $G \times_S X_{e,V}$ (intuitively $V \ni (g,x) \mapsto (g,(e,g^{-1}x)) \in G \times_S X_{e,V}$), let $\psi : V \to G \times_S X_{e,V}$ denote this map. It is easy to check that φ and ψ are inverse to each other and the multiplication map $G \times_S X_{e,V} \to X$ is identified with the second projection $\mathrm{pr}_2 : V \to X$. The rest of the lemma follows easily. $\square$

If $Y \subset T_K$ is a subvariety, compactify T_K with a projective space $\mathbf{P}_R$, let $\overline{Y}$ (resp. $\overline{Y}_K$) be the closure of Y in $\mathbf{P}_R$ (resp. $\mathbf{P}_K$). Note that $\overline{Y}$ is flat and proper over $\operatorname{Spec} R$, let P be the Hilbert polynomial of $\overline{Y}$. Thus $\overline{Y}$ corresponds to an R-point of $\mathrm{Hilb}_P(\mathbf{P}_R/R)$, i.e. $[\overline{Y}] \in \mathrm{Hilb}_P(\mathbf{P}_R/R)(R)$, with $[\overline{Y}_K]$ the generic point.

Proof of Proposition 3.1.11. Let X be the closure of T_K-orbit of $[\overline{Y}_K]$ in $\mathrm{Hilb}_P(\mathbf{P}_R/R)$, which is a (possibly non-normal) toric R-scheme. Let

$$\pi : \mathfrak{X} \to X$$

be the normalization. If we use the twisted action of T_R on $\mathrm{Hilb}_P(\mathbf{P}_R/R)$, i.e. $t \cdot [Z] = [t^{-1}Z]$, then $Y \subset T_K$ as the orbit in $\mathrm{Hilb}_P(\mathbf{P}_R/R)$ is identified with the visible contour in T_K, i.e $Y = \mathrm{Hilb}_{P,e}(\mathbf{P}_R/R) \cap T_K$, thus the closure of Y in X is $X \cap \mathrm{Hilb}_{P,e}(\mathbf{P}_R/R)$ which is also $X_{e,V}$ where $V = (T_R \times_R X) \cap \mathrm{Univ}\,|_X$. The closure $\overline{Y}$ of Y in $\mathfrak{X}$ is $\mathfrak{X}_{e,V'}$ where $V' = (T_R \times_R \mathfrak{X}) \cap \mathrm{Univ}\,|_{\mathfrak{X}} = \pi^*(V)$. Throw away orbits of $\mathfrak{X}$ that don't intersect $\overline{Y}$, and we still write $\mathfrak{X}$ for that toric scheme by abuse of notation. Then the structure map is surjective, and the flatness follows from Lafforgue transversality. □

3.2 Extension of results in constant coefficient

Lemma 3.2.1. *Let $Y \subset T_K$ be a subvariety, assume Y is geometrically integral, if Δ produces a tropical (resp. schön, resp. hübsch) compactification of Y, then $\Delta[d]$ also produces a tropical (resp. schön, resp. hübsch) compactification for $Y_{K'} \subset T_{K'}$ where $K' = K(t^{1/d})$.*

Proof. Let $R' = R[t^{1/d}]$, $\overline{Y}(\Delta[d])$ be the closure of $Y_{K'}$ in $\mathfrak{X}(\Delta[d])$, we have a fibre diagram (by lemma 3.1.9)

$$\begin{tikzcd} T_{R'} \times_{R'} \overline{Y}(\Delta[d]) \arrow[r] \arrow[d] & \mathfrak{X}(\Delta[d]) \arrow[d] \\ T_R \times_R \overline{Y} \arrow[r] & \mathfrak{X}(\Delta). \end{tikzcd}$$

Hence if $\overline{Y}$ is tropical (resp. schön), so is $\overline{Y}(\Delta[d])$.

If $\overline{Y}$ is hübsch, $\mathfrak{X}(\Delta)$ has reduced special fibre and so does $\mathfrak{X}(\Delta[d])$, thus $\overline{Y}(\Delta[d])$ and $\mathfrak{X}(\Delta[d])$ are obtained from $\overline{Y}$ and $\mathfrak{X}(\Delta)$ simply by ring extension $\operatorname{Spec} R' \to \operatorname{Spec} R$, therefore $\overline{Y}(\Delta[d])$ is also hübsch.

□

Proposition 3.2.2. *If Y is hübsch in T_K, assume Y is geometrically integral, then $\mathcal{T}(Y)$ has a minimal fan stucture Δ corresponding to the log canonical compactification.*

Proof. Let Δ' be another fan supported on $\mathcal{T}(Y)$, we show that Δ' is a refinement of Δ. Suppose on the contrary Δ' does not refine Δ, then there exists a d-dimensional cone $\sigma' \in \Delta'$ ($d = \dim Y$), which is not contained in any cone of Δ. There is a $(d-1)$-dimensional cone $\alpha \in \Delta$ which meets the interior of σ'. Let Δ'' be a common strictly simplicial refinement, there is a $(d-1)$-dimensional cone $\alpha'' \subset \alpha \cap \sigma'$, meeting the interior of both α and σ'. We may assume $\mathfrak{X}(\Delta), \mathfrak{X}(\Delta')$ and $\mathfrak{X}(\Delta'')$ all have reduced fibre, otherwise consider $\Delta[l], \Delta'[l]$ and $\Delta''[l]$ for some l (Lemma 3.2.1).

We have proper birational maps

$$p_1 : \mathfrak{X}'' := \mathfrak{X}(\Delta'') \to \mathfrak{X} := \mathfrak{X}(\Delta)$$

and

$$p_2 : \mathfrak{X}'' \to \mathfrak{X}' := \mathfrak{X}(\Delta').$$

Let Z, Z' and Z'' be the orbit closure V_α, $V_{\sigma'}$ and $V_{\alpha''}$ in $\mathfrak{X}$, $\mathfrak{X}'$ and $\mathfrak{X}''$ respectively.

Note that σ' is not contained in N since it's a maximal cone in $\mathcal{T}(Y)$, and consequently same for α and α'', hence Z, Z' and Z'' are all on the special fiber, in particular, are all normal toric varieties over k.

Z' is isomorphic to a torus T'_k. The induced morphism $p_2 : Z'' \to Z'$ is proper toric morphism of relative dimension 1, thus $Z'' \cong Z' \times_k \mathbb{P}^1_k$. $p_1 : Z'' \to Z$ is birational.

Let $\overline{Y}$ and $\overline{Y}''$ be the closure of Y in $\mathfrak{X}$ and $\mathfrak{X}''$ respectively. $\overline{Y}''$ is a schön compactification with reduced special fibre. The scheme-theoretic intersection $C := Y'' \cap Z'' \subset T'_k \times_k \mathbb{P}^1_k$ is 1-dimensional, reduced, proper and smooth over k, thus $C \cong z \times \mathbb{P}^1$ for some $z \in T'_k$ a 0-dimensional reduced closed subscheme. So we have that $K_C + B_C$ is trivial. Since $\overline{Y}'' \to \overline{Y}$ is log crepant, and $K_{\overline{Y}} + B_{\overline{Y}}$ is ample, by projection formula we conclude that C is contracted by $p_1 : Z'' \to Z$. Since p_1 is equivariant, all fibers $z' \times \mathbb{P}^1$ are contracted. A contradiction against that p_1 is a birational map. □

Theorem 3.2.3. *Assume* $\operatorname{char} K = 0$*, then for any variety Y over K, Y*

contains a schön very affine variety.

Proof. The proof proceeds in an analogous streamline as in the constant coefficient case , only with some technical issues to be taken care of. Let Y^o be a regular very affine variety over K and $\overline{Y}$ a regular compactification, projective over $\operatorname{Spec} R$ with reduced special fiber and simple normal crossing boundary divisor. Let M be a lattice with a homomorphism $\phi : M \to \mathcal{O}^*(Y)$ such that $K[M] \to \mathcal{O}(Y)$ is surjective. For any stratum S, let $\widetilde{M_S}$ be the set $\{\widetilde{m} = (m, r) \in \widetilde{M} = M \oplus \mathbb{Z} | \widetilde{m} \in \mathcal{O}(\mathrm{Star}(S))\}$ (where we think $\widetilde{m} = \phi(m)t^r$ as a rational function on Y), the conditions in proposition 1.4.6 are now the following:

1. for each strata S, $\mathrm{Star}\, M$ is affine and $A[M_S] \to \mathcal{O}(Star(S))$ is surjective.

2. for each strata S and any divisor $D_0 \in D_S$, there exists $\widetilde{m} \in \widetilde{M_S}$ such that $\mathrm{val}_{D_0} \widetilde{m} = 1$ and $\mathrm{val}_{D'} \widetilde{m} = 0$ for all $D' \in D_S \backslash \{D_0\}$.

3. the cones σ_S (the dual cone of $\widetilde{M_S}$) form an admissible fan in $\widetilde{N}_{\mathbb{Q}}$.

We may assume Y is regular, by Hironaka's resolution theorem and Mumford's semistable reduction theorem, there is a compactification $\overline{Y}$, possibly over a ring extension $R \subset R[t^{1/d}]$, which is regular with reduced special fibre and simple normal crossing boundary divisor. Thanks to the following lemma of a relative version of Bertini's theorem, the above conditions can be achieved by adding more generic hyperplane sections as in the proof of theorem 1.4.1. $\square$

Lemma 3.2.4 ([13])**.** *X is a regular scheme, flat and quasi-projective over* $\operatorname{Spec} R$*, assume X_s is reduced and simple normal crossing, then a general hyperplane $H \subset \mathbb{P}^n_R$ intersects X transversely and $(X \cap H) \cup X_s$ is simple normal crossing.*

Theorem 3.2.5. *If Y is schön in T_K, then any fan supported on $\mathcal{T}(Y)$ produces a schön compactification.*

Proof. This is a relative version of theorem 1.2.9, which is proved by reducing to the constant case. Let Δ' be any fan supported on $\mathcal{T}(Y)$, and Δ is a refinement of Δ' which is schön, let $\overline{Y}$ and $\overline{Y}'$ be the closure of Y in $\mathfrak{X}(\Delta)$ and $\mathfrak{X}(\Delta')$ respectively. We have a commutative diagram as follows:

$$\begin{tikzcd} T_R \times_R \overline{Y} \arrow[r, "f"] \arrow[d] & \mathfrak{X}(\Delta) \arrow[d, "p"] \\ T_R \times_R \overline{Y}' \arrow[r, "g"] & \mathfrak{X}(\Delta'). \end{tikzcd}$$

Note that f is smooth. Pull the diagram back to the generic fibre and lift to the algebraic closure $\mathbb{K}$, we have

$$\begin{tikzcd} T_{\mathbb{K}} \times_{\mathbb{K}} \overline{Y}_{\mathbb{K}} \arrow[r, "f_{\mathbb{K}}"] \arrow[d] & \mathfrak{X}(\Delta)_{\mathbb{K}} \arrow[d] \\ T_{\mathbb{K}} \times_{\mathbb{K}} \overline{Y}'_{\mathbb{K}} \arrow[r, "g_{\mathbb{K}}"] & \mathfrak{X}(\Delta')_{\mathbb{K}}. \end{tikzcd}$$

Now $\overline{Y}_{\mathbb{K}}$ is a disjoint union of irreducible components of pure dimension, each irreducible component has smooth surjective structure map possibly in a smaller toric open set (see remark 1.2.12). Apply theorem 1.3.5 to each irreducible component of $\overline{Y}_{\mathbb{K}}$, we see that $\overline{Y}_{\mathbb{K}}$ is a set theoretic inverse image of $\overline{Y}'_{\mathbb{K}}$, hence $\overline{Y}'_{\mathbb{K}}$ is also a disjoint union of irreducible components of pure dimension. We can now apply theorem 1.2.9 to each irreducible component of $Y_{\mathbb{K}}$, $g_{\mathbb{K}}$ is smooth.

Let W be any toric orbit closure of $\mathfrak{X}(\Delta)$ on the special fibre (which is a normal toric variety over k), let $O \subset W$ be the open orbit, then $W \to p(W)$ is a proper toric map. Restricting the diagram on W, we have

$$\begin{tikzcd} \overline{Y} \cap W \arrow[r] \arrow[d] & W \arrow[d, "p"] \\ \overline{Y}' \cap p(W) \arrow[r] & p(W). \end{tikzcd}$$

$\overline{Y} \cap W$ has smooth structure map in W, by the same argument as above, we can show that $\overline{Y}' \cap p(W)$ also has smooth structure map. Combining the above results, we see that f is smooth. □

Assume k is of characteristic 0, and $\mathbb{K}$ is the field of Puiseux series over k. For a subvariety $Y \subset T_{\mathbb{K}}$, Y is defined over $K_n := k((t^{1/n}))$ for some n. We propose the following definition.

Definition 3.2.6. $Y \subset T_K$ is schön (resp. hübsch) if for some n, Y is defined over K_n and $Y \subset T_{K_n}$ is schön (resp. hübsch).

3.2.7. It is possible to define a toric scheme over the valuation ring of $\mathbb{K}$ (as certain limit of toric schemes over discrete valuation rings), and define tropical compactifications using this toric schemes. The above definition is to avoid some technical difficulty such as schemes over a non-noetherian ring. It is clear that if $Y \subset T_{\mathbb{K}}$ is hübsch, $\mathcal{T}(Y)$ has a minimal fan structure.

Chapter 4

Some Examples

In this chapter we give some examples of schön and hübsch very affine varieties. The following are known examples:

1. smooth very affine curves,
2. hypersurfaces nondegenerate with respect to the Newton polytope (see [25]),
3. hyperplane complements (see section 4.1 and [8]),
4. moduli space of del Pezzo surfaces of degree 3 and 2 (see [9]),
5. open cubic surfaces (see section 4.2),
6. moduli space of 6 lines on $\mathbb{P}^2$ in linear general position (see [18] and [22] for the tropicalization),
7. moduli space of n-pointed curves of genus 0 (see [9]).

4.1 Linear subspaces and hyperplane complements

4.1.1. Let F be either k or K for constant or non-constant coefficient case respectively, in this section we study open linear subspaces of projective spaces. To be precise, X is a linear subspace of $\mathbb{P}_F^{n-1}$ defined over F, i.e. X is given by linear equations with coefficients in F, and $Y = X \cap T_F$. We assume Y is nonempty, and the matroid stucture associated to X (with the boundary divisors) is connected.

4.1.2. There is an equivalent way of saying this using hyperplane complements hyperplane complements. X is a projective space $\mathbb{P}_F^{r-1}$, and $H_1, \ldots, H_n$ are n hyperplanes of X given by linear equations $L_i = 0$ defined over F ($i = 1, \ldots, n$). Let Y be the complement. Assume also that the matroid structure associated to the hyperplanes is connected.

We have a closed embedding $X \to \mathbb{P}_F^{n-1}$ given by

$$x \mapsto [L_1 : \cdots : L_n].$$

In this way, we realize X as a linear subspace of $\mathbb{P}_F^{n-1}$ and Y is the intersection of X with the torus of $\mathbb{P}_F^{n-1}$.

4.1.3. Recall that a *matroid* structure $\mathcal{M}$ on n elements $[n] := \{1, 2, \ldots, n\}$ is a collection of subsets of $[n]$ with the following properties:

1. $\mathcal{M}$ is nonempty.

2. If $A \in \mathcal{M}$, then any subset of A is also in $\mathcal{M}$.

3. For any two $A, B \in \mathcal{M}$, if $|A| < |B|$, then there is some $p \in B \backslash A$, such that $A \cup \{p\} \in \mathcal{M}$.

4.1.4. The matroid associated to $H_1, \ldots, H_n$ is the collection of subsets

$$S \subseteq \{H_1, \ldots, H_n\}$$

such that the hyperplanes in S intersects in the expected codimension (codimension $= |S|$), or equivalently the hyperplanes in S are simple normal crossing. A matroid $\mathcal{M}$ is called *connected* if $\mathcal{M}$ is not a product of two matroids on A, B respectively, where $A \sqcup B = [n]$ is a partition.

Our main result in this section is the following.

Theorem 4.1.5. *Y is hübsch in either case.*

4.1.6. The constant coefficent case follows from results in [8]. We recall their construction since it is also useful in the non-constant coefficient case. $X \subset \mathbb{P}^{n-1}_k$ determines a point in the Grassmannian, $[X] \in G(r, n)$. Since T^{n-1} acts on $\mathbb{P}^{n-1}$, it also acts on the Grassmannian $G(r, n)$. Without loss of generality, we may assume that $[1 : 1 : \cdots : 1] \in X$ otherwise we just translate X by multiplication of some $t \in T^{n-1}$.

Let $\mathfrak{X}$ be the closure of the orbit $T^{n-1} \cdot [x]$ in the Grassmannian $G(r, n)$. Let $G_e(r, n)$ be the closed subvariety of $G(r, n)$ parametrizing all $(r-1)$-dimensional linear subspaces of $\mathbb{P}^{n-1}$ which contain the point $[1 : \cdots : 1]$. Let

$\overline{Y}$ be the scheme-theoretic intersection of $\mathfrak{X}$ and $G_e(r,n)$ in $G(r,n)$. Then we have the following results.

Theorem 4.1.7 (White, [26])**.** *$\mathfrak{X}$ is a normal toric variety. In fact it is projectively normal with the embedding in the Plücker coordinates.*

Theorem 4.1.8 (Hacking-Keel-Tevelev, [8])**.** *The structure map $T^{n-1} \times \overline{Y} \to \mathfrak{X}$ is smooth and the log canonical divisor $K_{\overline{Y}} + B_{\overline{Y}}$ is very ample.*

4.1.9. Therefore we conclude that Y (in the constant coefficient case) is hübsch and the log canonical compactification is given by $\mathfrak{X} \cap G_e(r,n)$. To describe the tropicalization of Y, we need introduce more concepts of matroid theory. If $\mathcal{M}$ is a matroid on $[n]$, maximal elements in $\mathcal{M}$ are called *bases* of $\mathcal{M}$. They all have same cardinality, called the *rank* of $\mathcal{M}$. In our case, the rank for the matroid of $H_1, \ldots, H_n$ on $X \simeq \mathbb{P}^{r-1}$ is r. A subset

$$S \subset \{H_1, \ldots, H_n\}$$

is a base if and only if the intersection of the elements in S is empty.

4.1.10. Given a matroid $\mathcal{M}$, we can construct a polytope associated to $\mathcal{M}$, called *matroid polytope*, denoted by $P_{\mathcal{M}}$. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$, for each base $S \in \mathcal{M}$, we draw a lattice point equal to $\sum_{i \in S} e_i$. The matroid polytope associated to $\mathcal{M}$ is the convex hull of these lattice points in $\mathbb{R}^n$. The matroid polytope is contained in the hyperplane

$$x_1 + \cdots + x_n = r.$$

4.1.11. The universal matroid $U(r, n)$ of rank r is the matroid structure on $[n]$ which is consisted of all subsets of $[n]$ with r or fewer elements. The matroid polytope associated to the universal matroid $U(r, n)$ is call the *hypersimplex* $\Delta(r, n)$, which is the convex hull of all $\sum_{i \in I} e_i$ for $I \subset [n]$ and $|I| = n$. In general, for a matroid $\mathcal{M}$ of rank r, the matroid polytope $P_{\mathcal{M}}$ is contained in $\Delta(r, n)$. The following is a well-known result.

Theorem 4.1.12 ([6])**.** *An edge of $\mathcal{M}$ is also an edge of $\Delta(r, n)$, it is of the form $e_i - e_j$. If the convex hull of a collection of vertices of $\Delta(r, n)$ has this property, then it is a matroid polytope, i.e. it is the matroid polytope associated to some matroid of rank r.*

4.1.13. In our case, X lives in a projective space $\mathbb{P}^{n-1}$ as a linear subspace, and the hyperplanes on X are given as intersection of coordinate hyperplanes, i.e. $H_i = X \cap (x_i = 0)$, then $I \subset [n]$ is a base if and only if the Plücker coordinate of $[X]$ index by I is nonzero. Thus the polytope defining the toric variety of $\overline{T \cdot [X]}$ is exactly the matroid polytope associated to the matroid of $H_1, \ldots, H_n$.

4.1.14. White's result (Theorem 4.1.7) is in fact a combinatorial result about matroid polytopes. That is if $P \subset \Delta(r, n)$ is a matroid polytope, then any lattice point in rP can be written as a sum of r lattice points in P. From the theory of tropical compactification, we see that the tropicalization of Y is a subfan of the normal fan of $P_{\mathcal{M}}$ consisted of cones whose corresponding orbits in $\overline{T \cdot [X]}$ intersect $\overline{Y}$. However, to determine which cones are in the tropicalization is not an easy task.

Proof of Theorem 4.1.5. We only need to prove the non-constant coefficient case. When the matroid is connected, Y is regid. If $X \subset \mathbb{P}^{n-1}_K$ is a linear subspace of dimension $r-1$ defined over K, it determines a K-point $[X] \in G(r,n)(K)$. Without loss of generality we may assume $e = [1 : \cdots : 1] \in X$. Since T^{n-1} acts on $\mathbb{P}^{n-1}$, it also acts on $G(r,n)$. Let $\mathfrak{X}$ be the closure of the orbit $T^{n-1}_K \cdot [X]$ in $G(r,n) \times \operatorname{Spec} R$. By lemma below, $\mathfrak{X}$ is a normal toric R-scheme.

This is exactly the construction in the existence of tropical compactification since now the irreducible component of the Hilbert scheme containing $[X]$ is $G(r,n) \times \operatorname{Spec} R$. Thus $\mathfrak{X} \cap G_e(r,n)$ is a tropical compactification. Furthermore, by Lafforgue transversality, the structure map $T_R \times_R \overline{Y} \to \mathfrak{X}$ is identified with the projection $\mathrm{Univ}|_{\mathfrak{X}} \cap T \to \mathfrak{X}$, which is smooth.

The log canonical bundle

$$K_{\overline{Y}} + B_{\overline{Y}} = \det \mathscr{N}_{\overline{Y}/\mathfrak{X}} = \det(\mathscr{N}_{G_e/G})|_{\overline{Y}}$$

is the restriction of the tautological line bundle, hence it is ample. Y is hübsch.

□

Lemma 4.1.15. *The closure of a T^{n-1}_K-orbit of a K-point of $G(r,n)$ in $G(r,n) \times$ $\operatorname{Spec} R$ is a normal toric R-scheme, in fact it is projectively normal in the Plücker embedding.*

Proof. Let $x \in G(r,n)(K)$ be a K-point of the Grassmannian, with Plücker coordinates $x = (x_I)$. The bases of the matroid structure associated to x is the set $\mathcal{M} = \{I | x_I \neq 0\}$. Let $\mathfrak{X}$ be the closure of $T_K^{n-1} \cdot x$ in $G(r,n) \times \operatorname{Spec} R$. By the discussion in 2.3.9, we see that $\mathfrak{X}$ is projectively normal if and only if for any $r > 0$, a lattice point in $rP_{\mathcal{A}}$ can be written as a sum of r lattice points in $P_{\mathcal{A}}$ and some nonnegative multiple of e, where $\mathcal{A} = \{(e_I, v(x_I) | x_I \neq 0\} \subset \mathbb{Z}^n \oplus \mathbb{Z}$, and $P_{\mathcal{A}}$ is the admissible convex hull.

We note that the heights $v(x_I)$ satisfies tropical Plücker relation (see [21], 4.2.1), the projection of the lower faces of $P_{\mathcal{A}}$ on to $P_{\mathcal{M}}$ form a matroid subdivision of the matroid polytope $P_{\mathcal{M}}$, i.e.

$$P_{\mathcal{M}} = \coprod_i P_{\mathcal{M}_i},$$

where $P_{\mathcal{M}_i}$ are submatroid polytopes of $P_{\mathcal{M}}$.

For any lattice point $y = (y_0, y_1) \in rP_{\mathcal{A}}$ with $y_0 \in \mathbb{Z}^n$ and $y_1 \in \mathbb{Z}$, we have $y_0 \in rP_{\mathcal{M}_i}$ for some $\mathcal{M}_i$. By Theorem 4.1.7, y_0 can be written as a sum of r lattice point in $P_{\mathcal{M}_i}$, say

$$y_0 = e_{I_1} + \cdots + e_{I_r}.$$

Since $(e_{I_i}, v(x_{I_i}))$ is on the lower boundary of $P_{\mathcal{A}}$, we have

$$y_1 \geq v(x_{I_1}) + \cdots v(x_{I_r}).$$

The statement follows. $\square$

Remark 4.1.16. In [8], the authors constucted a modular compactification of the moduli space hyperplane arrangements in linear general position. If Y (over $\operatorname{Spec} K$) is a hyperplane complement in general position, an easier way to see that Y is hübsch is to take the K point in the moduli space and take it closure in the compactification. Our method works for any hyperplane complement (with connected matroid structure).

4.2 Open del Pezzo surfaces

4.2.1. Let X be a smooth del Pezzo surface of degree $9-n$ for $n = 3, 4, \ldots, 8$, that is a nonsingular projective surface X with $-K_X$ ample. It is known from classical algebraic geometry that X is obtained from $\mathbb{P}^2$ by blowing up n distinct points in general position. General position here means no 3 points are on a line, no 6 points are on a conic and when $n = 8$, the 8 points do not lie on a singular cubic with one of the points located at the singular point. Any realization $X \to \mathbb{P}^2$ is called a blow-up model.

4.2.2. Let $Y \subset X$ be the complement of (-1)-curves on a cubic surface X. Clearly, given a blow-up model, all the (-1)-curves on X are exactly strict transform of lines passing through 2 points, conics passing through 5 points, and exceptional divisors. Y is very affine. The main result of this section is the following.

Theorem 4.2.3. *If X is a cubic surface, then Y is hübsch, and the log canonical compactification of Y is X with all Eckhart points blown-up (if there are any).*

4.2.4. An *Eckhart* point of a cubic surface is an ordinary triple point of the 27 (-1)-curves. We give some examples of Eckhart points on a cubic surface.

1. Case 1: Let $X \to \mathbb{P}^2$ be a blow-up model for a cubic surface, $P_1, \ldots, P_6$ be the six points that are blown up. If three lines P_1P_2, P_3P_4 and P_5P_6 meet at a point E, then E is an Eckhart point.

2. Case 2: Again, given a blow-up model $X \to \mathbb{P}^2$, if the line P_1P_6 is tangent to the conic passing through all P_i except P_6, then L_{16}, E_1, C_6 meet at an Eckhart point, where L_{16} is the strict transform of P_1P_6, E_1 is the exceptional divisor over P_1 and C_6 is the strict transform of the conic passing through all P_i except P_6.

Lemma 4.2.5. *If X is a cubic surface, any Eckhart point can be realized as one of the above two cases after choosing an appropriate blow-up model, moreover even the above two cases are equivalent.*

4.2.6. We review some results on the symmetries of cubic surfaces. If $X \to \mathbb{P}^2$ is a blown-up of 6 points $P_1, \ldots, P_6$, let E_i $(i = 1, \ldots, 6)$ be the exceptional divisor over P_i, L_{ij} $(1 \le i < j \le 6)$ be the strict transform of P_iP_j, C_j $(j = 1, \ldots, 6)$ be the strict transform of the conic passing through all P_i except P_j.

A marking of the 27 (-1)-curves with E_i, L_{ij}, C_j satisfying the incidence relation determines a blow-up model. It is a classical result that the automorphism group of the 27 (-1)-curves preserving the incidence relation is $W(E_6)$, the Weyl group associated to the root system E_6.

One way to get from one blow-up model to another is by Cremona transformation. Recall that a Cromona transform is a birational automorphism of $\mathbb{P}^2$ given by blowing up 3 points A, B, C in linear general position and then blowing down the strict transform of AB, BC, CA. If we choose coordinates on $\mathbb{P}^2$ such that the coordinates of A, B, C are $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$ respectively, then the Cremona transformation is given by

$$[x:y:z] \mapsto [yz:xz:xy],$$

defined everywhere except A, B, C.

If $X \to \mathbb{P}^2$ is a cubic surface given by blowing up $P_1, \dots, P_6$, performing a Cremona transformation at any three distinct P_i, P_j, P_k will produce another blow-up model. Say we perform a Cremona tranformation at P_1, P_2, P_3, then the new blow-up model $X \to \mathbb{P}^2$ is blowing-up $P_1', P_2', P_3', P_4, P_5, P_6$, where P_1' is the blown-down of L_{23}, similarly for P_2' and P_3'. In this case we get an automorphism of the 27 (-1)-curves explicitly as follows:

1. $E_i \mapsto L_{jk}$, $L_{ij} \mapsto E_k$, $C_i \mapsto C_i$ for $\{i, j, k\} = \{1, 2, 3\}$.

2. $E_i \mapsto E_i$, $L_{ij} \mapsto C_k$, $C_i \mapsto L_{jk}$ for $\{i, j, k\} = \{4, 5, 6\}$

3. $L_{ij} \mapsto L_{ij}$ for $i \in \{1, 2, 3\}$ and $j \in \{4, 5, 6\}$.

It is known that the Cremona tranformations and S_6 generate the Weyl group $W(E_6)$.

Proof of Lemma 4.2.5. Let E be an Eckhart point on a cubic surface $X \to \mathbb{P}^2$

for some blow-up model. If one of the 3 lines passing through E is labelled E_i, then on $\mathbb{P}^2$, the only possibility is case 2. We can always make one of the lines labelled by E_i as we see from the explicit automorphism of the 27 lines above.

If E is the intersection of E_1, L_{16} and C_6, we perform a Cremona transform at P_1, P_2, P_3, E is then the intersection of L_{23}, L_{16} and L_{45}, which is the first case. $\square$

4.2.7. We shall describe the tropicalization of Y. Suppose first that there is no Eckhart point, then X is a smooth compactification of Y with simple normal crossing boundary divisors. To compute $M_Y = \mathcal{O}^*(Y)/k^*$, note that Y is the complement of $\mathbb{P}^2$ of 15 lines and 6 conics, thus we have an exact sequence

$$0 \to M_Y \to \mathbb{Z}^{21} \to \operatorname{Pic} \mathbb{P}^2 \to 0.$$

We see that $M_Y \cong \mathbb{Z}^{20}$. Each boudary divisor D of X determines a valuation $\mathrm{val}_D \in \mathrm{Hom}_{\mathbb{Z}}(M_Y, \mathbb{Z})$, which belongs to $\mathrm{trop}(Y)$. By geometric tropicalization, val_D and $\mathrm{val}_{D'}$ span a cone in $\mathrm{trop}(Y)$ if and only if D and D' intersects. Thus the link of $\mathrm{trop}(Y)$ is exactly the dual graph of the boundary divisors.

The appearence of Eckhart points does not change M_Y and $\mathrm{val}_D \in M_Y$, however it changes the tropicalization of Y since now $X \backslash Y$ is not simple normal crossing. Let $\widetilde{X} \to X$ be the blow-up of all Eckhart points, $\widetilde{X} \backslash Y$ is simple normal crossing, thus we should modify $\mathrm{trop}(Y)$ in the following way, whenever D_1, D_2, D_3 meets in an Eckhart point, we shall introduce a new lattice point

$v = \mathrm{val}_{D_1} + \mathrm{val}_{D_2} + \mathrm{val}_{D_3}$ in N_Y, and replace the cones

$$\langle \mathrm{val}_{D_1}, \mathrm{val}_{D_2} \rangle, \langle \mathrm{val}_{D_2}, \mathrm{val}_{D_3} \rangle, \langle \mathrm{val}_{D_1}, \mathrm{val}_{D_3} \rangle$$

by the cones

$$\langle v, \mathrm{val}_{D_1} \rangle, \langle v, \mathrm{val}_{D_2} \rangle, \langle v, \mathrm{val}_{D_3} \rangle.$$

Proof of Theorem 4.2.3. We first show that $Y \subset \widetilde{X}$ satisfies all the conditions in Proposition 1.4.6. We will call the strict transform of 27 lines in $\widetilde{X}$ still the 27 lines by abuse of language, and exceptional divisors over Eckhart points the new divisors. Let S be a stratum, there are 4 cases to consider.

1. If $\dim S = 1$ and S is contained in the 27 lines, after an appropriate labelling, we may assume $S \subset L_{12}$ in X, thus $\mathrm{Star}(S)$ equals $\mathbb{P}^2$ minors 14 lines and 6 conics, which is very affine.

2. If $\dim S = 1$ and S is contained in the new divisors, without loss of generality, we may assume S is contained in the exceptional divisor over E, intersection of P_1P_2, P_3P_4, P_5P_6. After Cremona transformation at E, P_1, P_3, we see that $\mathrm{Star}(S)$ equals $\mathbb{P}^2$ minors 13 lines and 6 conics, which is very affine.

3. If $\dim S = 0$ and S is not contained in the new divisors, after an appropriate Cremona transformation, we may assume none of the two divisors that contain S is E_i, then they are either two lines or one line and one conic or two conics. In either case, $\mathrm{Star}(S)$ is $\mathbb{P}^2$ minors some (at least

13) lines and some conics, which is very affine.

4. If $\dim S = 0$ and S is contained in a new divisor, without loss of generality, we may assume E (the Eckhart point) is the intersection of P_1P_2, P_3P_4, P_5P_6, and S is the intersection of L_12 and F (the exceptional divisor over E). Perform Cremona transformation at E, P_3, P_5, we see $\mathrm{Star}(S)$ is very affine.

To verify condition (2) in theorem 1.4.6, we only need to consider case 3 and 4 above. The statement is obvious if we can illustrate a line not passing through S. In case 3, we may assume S is the intersection of L_{12} with L_{13}, or L_{12} with C_2 or C_3, or C_4 with C_5, in either case, L_{45} is such a line. In case 4, P_2P_3 is such a line.

To verify condition (3), let S, S' be two distinct strata. We may assume S and S' are 0-dimensional. If $D_S \cap D_{S'}$ is nonempty, assume that

$$D_S = \{D, E\}, D_{S'} = \{E, F\}.$$

If E is a new divisor, then after blowing down E, the image of D, F are two lines, clearly there is a unit $u \in M_Y$ such that $(u) = D - F$. If E is one of the 27 line, we may choose a blow-up model such that E is of the form L_{12}.

Consider the map $\pi : \widetilde{X} \to \mathbb{P}^2$. The image of E is the line P_1P_2, the image of D is either P_1 or a line or a conic passing through P_1, and similarly for the image of F. We may use the line P_3P_4 to get a unit $u \in M_Y$ with $\mathrm{val}_D\, u < 0, \mathrm{val}_E\, u = 0$ and $\mathrm{val}_F\, u > 0$.

If the image of D is P_1, then we use the unit u_1 such that $(u_1) = P_3P_4 - P_3P_1$, then $\operatorname{val}_D u_1 < 0$ and $\operatorname{val}_E u_1 = 0$. If the image of D is a line or a conic through P_1, we use the unit u_1 such that $(u_1) = P_1P_2 - \pi(D)$ or $2P_1P_2 - \pi(D)$, then $\operatorname{val}_D u_1 < 0$ and $\operatorname{val}_E u_1 = 0$. Similarly for F, we can always find u_2 such that $\operatorname{val}_F u_2 > 0$ and $\operatorname{val}_E u_2 = 0$. Therefore $u = u_1u_2$ will work.

Finally we need to consider the case when $D_S \cap D_{S'} = \emptyset$. This can be done following the same idea. Consider any blow-up model $\pi : \widetilde{X} \to \mathbb{P}^2$ such that $\pi(S) \neq \pi(S')$. This can always be achieved, if $\pi(S) = \pi(S') = P_1$, then we change a blow-up model such that the exceptional divisor E_1 is changed to a line. Without loss of generality, assume $\pi(S) = P_1$ and $\pi(S') = P_2$, again we can use line P_3P_4 to obtain a unit u with positive valuations on divisors in D_S and negative valuations on divisors in $D_{S'}$. We omit tedious details here.

Now we have showed that $\widetilde{X}$ with the boundary divisors satisfies the conditions in 1.4.6, therefore Y is schön, and a fan structure obtained from $\{\sigma_S\}$ gives exactly $\widetilde{X}$. This fan clearly is a minimal fan strucure since $\operatorname{trop}(Y)$ is 2-dimensional and each ray is a common face for at least three cones. Each 1-dimensional stratum is log mimimal since it is $\mathbb{P}^1$ minors at least 3 points. This verifies that Y is hübsch and the log canonical model for Y is $\widetilde{X}$.

$\square$

Remark 4.2.8. If we notice that $\operatorname{Pic} X$ is a free abelian group and is generated by boundary divisors, it is easier to verify these conditions in Proposition 1.4.6. For example to check condition (2), we only check that $\operatorname{Pic} X$ is generated by boundary divisors not containing S for any stratum S as pointed out in Remark 1.4.8. And for condition (1), we can now check the new condition (1) (see 1.4.8), i.e. each stratum S is very affine, and $M^S \to \mathcal{O}(S)/k^*$ is surjective.

4.2.9. This result is not true for del Pezzo surfaces of degree 2. Here is an example. Let $X \to \mathbb{P}^2$ be blow-up 7 point $P_1, \ldots, P_7$ on $\mathbb{P}^2$. Suppose the line P_1P_2 is tangent to the conic passing through the other five points, and the tangent point is not one of the other five points. Then the complement of (-1)-curves on X is not schön.

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